Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
C. R. Acad. Sci. Paris, Ser. I 339 (2004) 417–420
COMPTES RENDUS MATHEMATIQUE

Differential Topology

Isotropic nonarchimedean S-arithmetic groups are not left orderable

Lucy Lifschitz [a], Dave Witte Morris [b,c,*]

[a] *Department of Mathematics, University of Oklahoma, Norman, OK 73019, USA*
[b] *Department of Mathematics and Computer Science, University of Lethbridge, Lethbridge, Alberta T1K 3M4, Canada*
[c] *Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, USA*

Received 30 May 2004; accepted after revision 6 July 2004

Presented by Étienne Ghys

Abstract

If $\mathcal{O}_S$ is the ring of S-integers of an algebraic number field F, and $\mathcal{O}_S$ has infinitely many units, we show that no finite-index subgroup of $\mathrm{SL}(2, \mathcal{O}_S)$ is left orderable. (Equivalently, these subgroups have no nontrivial orientation-preserving actions on the real line.) This implies that if G is an isotropic F-simple algebraic group over an algebraic number field F, then no nonarchimedean S-arithmetic subgroup of G is left orderable. Our proofs are based on the fact, proved by D. Carter, G. Keller, and E. Paige, that every element of $\mathrm{SL}(2, \mathcal{O}_S)$ is a product of a bounded number of elementary matrices. ***To cite this article: L. Lifschitz, D.W. Morris, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Les groupes S-arithmétiques non-archimédiens isotropes ne sont pas ordonnables à gauche. Si $\mathcal{O}_S$ est l'anneau des S-entiers d'un corps de nombres F, et $\mathcal{O}_S$ a une infinité d'unités, nous prouvons qu'aucun sous-groupe d'indice fini de $\mathrm{SL}(2, \mathcal{O}_S)$ n'est ordonnable à gauche. (En d'autres termes, les sous-groupes d'indice fini de $\mathrm{SL}(2, \mathcal{O}_S)$ ne possèdent pas d'action non triviale sur la droite réelle respectant l'orientation.) Cela implique que si G est un groupe algébrique F-simple isotrope, défini sur un corps de nombres F, alors aucun sous-groupe S-arithmétique non-archimédien de G n'est ordonnable à gauche. La démonstration est fondée sur le fait, dû à D. Carter, G. Keller, et E. Paige, que chaque élément de $\mathrm{SL}(2, \mathcal{O}_S)$ est le produit d'un nombre borné de matrices élémentaires. ***Pour citer cet article : L. Lifschitz, D.W. Morris, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

* Corresponding author.
E-mail addresses: LLifschitz@math.ou.edu (L. Lifschitz), Dave.Morris@uleth.ca (D.W. Morris).
URLs: http://www.math.ou.edu/~llifschitz/ (L. Lifschitz), http://people.uleth.ca/~dave.morris/ (D.W. Morris).

1631-073X/$ – see front matter © 2004 Académie des sciences. Published by Elsevier SAS. All rights reserved.
doi:10.1016/j.crma.2004.07.015

1. Introduction

It is known [9] that finite-index subgroups of $\mathrm{SL}(3,\mathbb{Z})$ or $\mathrm{Sp}(4,\mathbb{Z})$ are not left orderable. (That is, there does not exist a total order $\prec$ on any finite-index subgroup, such that $ab \prec ac$ whenever $b \prec c$.) More generally, if G is a $\mathbb{Q}$-simple algebraic $\mathbb{Q}$-group, with $\mathbb{Q}\text{-rank}\, G \geqslant 2$, then no finite-index subgroup of $G_{\mathbb{Z}}$ is left orderable. It has been conjectured that the restriction on $\mathbb{Q}$-rank can be replaced with the same restriction on $\mathbb{R}$-rank, which is a much weaker hypothesis:

Conjecture 1. *If G is a $\mathbb{Q}$-simple algebraic $\mathbb{Q}$-group, with $\mathbb{R}\text{-rank}\, G \geqslant 2$, then no finite-index subgroup Γ of $G_{\mathbb{Z}}$ is left orderable.*

In other words, if H is a connected, semisimple real Lie group, with $\mathbb{R}\text{-rank}\, H \geqslant 2$, and Γ is an irreducible lattice in H, then Γ is not left orderable.

It is natural to propose an analogous conjecture that replaces $\mathbb{Z}$ with a ring of S-integers, and weakens the restriction on $\mathbb{R}$-rank. For simplicity, let us state it only in the case where $\mathbb{R}\text{-rank}\, G \geqslant 1$.

Conjecture 2. *If G is a $\mathbb{Q}$-simple algebraic $\mathbb{Q}$-group, with $\mathbb{R}\text{-rank}\, G \geqslant 1$, and $\{p_1, \ldots, p_n\}$ is any nonempty set of prime numbers, then no finite-index subgroup Γ of $G_{\mathbb{Z}[1/p_1,\ldots,1/p_n]}$ is left orderable.*

In other words, if H is a product of noncompact real and p-adic simple Lie groups, with at least one real factor and at least one p-adic factor, and Γ is any irreducible lattice in H, then Γ is not left orderable.

We prove Conjecture 2 under the additional assumption that $\mathbb{Q}\text{-rank}\, G \geqslant 1$:

Theorem 1.1. *If G is a $\mathbb{Q}$-simple algebraic $\mathbb{Q}$-group, with $\mathbb{Q}\text{-rank}\, G \geqslant 1$, and $\{p_1, \ldots, p_n\}$ is any nonempty set of prime numbers, then no finite-index subgroup Γ of $G_{\mathbb{Z}[1/p_1,\ldots,1/p_n]}$ is left orderable.*

More generally, if H is a product of real and p-adic simple Lie groups, with at least one p-adic factor, and Γ is any irreducible lattice in H, such that H/Γ is not compact, then Γ is not left orderable.

We also prove some cases of Conjecture 1 (with $\mathbb{Q}\text{-rank}\, G = 1$). For example, we consider the case where every simple factor of $G_{\mathbb{R}}$ (or of H) is isomorphic to $\mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SL}(2,\mathbb{C})$:

Theorem 1.2. *If $\mathcal{O}$ is the ring of integers of a number field F, and F is neither $\mathbb{Q}$ nor an imaginary quadratic extension of $\mathbb{Q}$, then no finite-index subgroup Γ of $\mathrm{SL}(2,\mathcal{O})$ is left orderable.*

In geometric terms, the theorems can be restated as the nonexistence of orientation-preserving actions on the line:

Corollary 1.3. *If Γ is as described in Theorem 1.1 or Theorem 1.2, then there does not exist any nontrivial homomorphism $\varphi : \Gamma \to \mathrm{Homeo}^+(\mathbb{R})$.*

Combining this corollary with an important theorem of Ghys [4] yields the conclusion that every orientation-preserving action of Γ on the circle S^1 is of an obvious type; any such action is either virtually trivial or semiconjugate to an action by linear-fractional transformations, obtained from a composition $\Gamma \to \mathrm{PSL}(2,\mathbb{R}) \hookrightarrow \mathrm{Homeo}^+(S^1)$. See [5] for a discussion of the general topic of group actions on the circle.

It has recently been proved that certain individual arithmetic groups are not left orderable (see, e.g., [3]), but our results apparently provide the first new examples in more than ten years of arithmetic groups that have no left-orderable subgroups of finite index. They are also the only known such examples that have $\mathbb{Q}$-rank 1.

If Γ is as described in Theorem 1.1 or Theorem 1.2, then Γ contains a finite-index subgroup of $\mathrm{SL}(2,\mathcal{O}_S)$, where S is a finite set of places of some algebraic number field F (containing all the archimedean places), such that the corresponding ring $\mathcal{O}_S$ of S-integers has infinitely many units. The theorems are obtained by reducing to the fact, proved by Carter, Keller, and Paige [1], that $\mathrm{SL}(2, \mathcal{O}_S)$ has bounded generation by unipotent elements. (That is, the fact that $\mathrm{SL}(2, \mathcal{O}_S)$ is the product of finitely many of its unipotent subgroups. See [7] for a recent discussion of bounded generation. Partial results were proved previously in [2] and [6].) We are also able to prove this reduction for noncocompact lattices in $\mathrm{SL}(3, \mathbb{R})$:

Theorem 1.4. *Suppose Γ is a finite-index subgroup of either*

(i) $\mathrm{SL}(2, \mathbb{Z}[1/r])$, *for some natural number $r > 1$, or, more generally,*
(ii) $\mathrm{SL}(2, \mathcal{O}_S)$, *where S is a finite set of places of an algebraic number field F (containing all the archimedean places), such that the corresponding ring $\mathcal{O}_S$ of S-integers has infinitely many units, or*
(iii) *an arithmetic subgroup of a quasi-split $\mathbb{Q}$-form of the $\mathbb{R}$-algebraic group* $\mathrm{SL}(3, \mathbb{R})$.

If $\varphi : \Gamma \to \mathrm{Homeo}^+(\mathbb{R})$ is any homomorphism, and U is any unipotent subgroup of Γ, then every $\varphi(U)$-orbit on $\mathbb{R}$ is bounded.

Corollary 1.5. *Suppose*

- *Γ is as described in Theorem 1.4, and*
- *Γ is commensurable to a group that has bounded generation by unipotent elements.*

Then every homomorphism $\varphi : \Gamma \to \mathrm{Homeo}^+(\mathbb{R})$ is trivial. Therefore, Γ is not left orderable.

2. Proof of Theorem 1.4(i)

Notation 1. *For convenience, let*

$$\bar{u} = \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}, \qquad \underline{v} = \begin{bmatrix} 1 & 0 \\ v & 1 \end{bmatrix}, \qquad \hat{s} = \begin{bmatrix} s & 0 \\ 0 & 1/s \end{bmatrix}$$

for $u, v \in \mathbb{Z}[1/r]$ and $s \in \{ r^n \mid n \in \mathbb{Z} \}$.

Suppose some $\varphi(U)$-orbit on $\mathbb{R}$ is not bounded above. (This will lead to a contradiction.) Let us assume U is a maximal unipotent subgroup of Γ.

Let V be a subgroup of Γ that is conjugate to U, but is not commensurable to U. Then $V_{\mathbb{Q}} \neq U_{\mathbb{Q}}$. Because $\mathbb{Q}$-rank $\mathrm{SL}(2, \mathbb{Q}) = 1$, this implies that $V_{\mathbb{Q}}$ is opposite to $U_{\mathbb{Q}}$. Therefore, after replacing U and V by a conjugate under $\mathrm{SL}(2, \mathbb{Q})$, we may assume

$$U = \{ \bar{u} \mid u \in \mathbb{Z}[1/r] \} \cap \Gamma \quad \text{and} \quad V = \{ \underline{v} \mid v \in \mathbb{Z}[1/r] \} \cap \Gamma.$$

Because V is conjugate to U, we know that some $\varphi(V)$-orbit is not bounded above. Let

$$x_U = \sup\{ x \in \mathbb{R} \mid \text{the } \varphi(U)\text{-orbit of } x \text{ is bounded above} \} < \infty \quad \text{and}$$
$$x_V = \sup\{ x \in \mathbb{R} \mid \text{the } \varphi(V)\text{-orbit of } x \text{ is bounded above} \} < \infty.$$

Assume, without loss of generality, that $x_U \geqslant x_V$.

Fix some $s = r^n > 1$, such that $\hat{s} \in \Gamma$, and let $B = \langle \hat{s} \rangle U$. Because $\langle \hat{s} \rangle$ normalizes U, this is a subgroup of Γ. Note that $\varphi(B)$ fixes x_U, so it acts on the interval (x_U, ∞). Since $\varphi(B)$ is nonabelian, it is well known (see, e.g., [5, Thm. 6.10]) that some nontrivial element of $\varphi(B)$ must fix some point of (x_U, ∞). In fact, it is not difficult to see that each element of $\varphi(B) \setminus \varphi(U)$ fixes some point of (x_U, ∞). In particular, $\varphi(\hat{s})$ fixes some point x of (x_U, ∞).

The left-ordering of any additive subgroup of $\mathbb{Q}$ is unique (up to a sign), so we may assume that

$$\varphi(\overline{u_1})x < \varphi(\overline{u_2})x \Leftrightarrow u_1 < u_2 \quad \text{and} \quad \varphi(\underline{v_1})x < \varphi(\underline{v_2})x \Leftrightarrow v_1 < v_2.$$

The $\varphi(U)$-orbit of x is not bounded above (because $x > x_U$), so we may fix some $u_0, v_0 > 0$, such that

$$\varphi(\underline{v_0})x < \varphi(\overline{u_0})x.$$

For any $\underline{v} \in V$, there is some $k \in \mathbb{Z}^+$, such that $v < s^{2k}v_0$. Then, because $\varphi(\hat{s})$ fixes x and $s^{-2k} < 1$, we have

$$\begin{aligned}\varphi(\underline{v})x &< \varphi(\underline{s^{2k}v_0})x = \varphi(\hat{s}^{-k}\underline{v_0}\hat{s}^k)x = \varphi(\hat{s}^{-k})\varphi(\underline{v_0})x \\ &< \varphi(\hat{s}^{-k})\varphi(\overline{u_0})x = \varphi(\hat{s}^{-k}\overline{u_0}\hat{s}^k)x = \varphi(\overline{s^{-2k}u_0})x < \varphi(\overline{u_0})x.\end{aligned}$$

So the $\varphi(V)$-orbit of x is bounded above by $\varphi(\overline{u_0})x$. This contradicts the fact that $x > x_U \geqslant x_V$.

3. Other parts of Theorem 1.4

(ii) The above proof of case (i) needs only minor modifications to be applied with a more general ring $\mathcal{O}_S$ of S-integers in the place of $\mathbb{Z}[1/r]$. (We choose $s = \omega^n$, where ω is a unit of infinite order in $\mathcal{O}_S$.) The one substantial difference between the two cases is that the left-ordering of the additive group of $\mathcal{O}_S$ is far from unique—there are usually infinitely many different orderings. Fortunately, we are interested only in left-orderings of $U = \{\overline{u} \mid u \in \mathcal{O}\} \cap \Gamma$ that arise from an unbounded $\varphi(U)$-orbit, and it turns out that any such left-ordering must be invariant under conjugation by $\hat{s}$. The left-ordering must, therefore, arise from a field embedding σ of F in $\mathbb{C}$ (such that $\sigma(s)$ is real whenever $\hat{s} \in \Gamma$), and there are only finitely many such embeddings. Hence, we may replace U and V with two conjugates of U whose left-orderings come from the same field embedding (and the same choice of sign).

(iii) A serious difficulty prevents us from applying the above proof to quasi-split $\mathbb{Q}$-forms of $\mathrm{SL}(3,\mathbb{R})$. Namely, the reason we were able to obtain a contradiction is that if $\overline{u_0}$ is upper triangular, $\underline{v}$ is lower triangular, $\hat{s}$ is diagonal, and $\lim_{k\to\infty} \hat{s}^{-k}\overline{u_0}\hat{s}^k = \infty$ under an ordering of Γ, then $\lim_{k\to\infty} \hat{s}^{-k}\underline{v}\hat{s}^k = e$. Unfortunately, the "opposition involution" of $\mathrm{SL}(3,\mathbb{R})$ causes the calculation to result in a different conclusion in case (iii): if $\hat{s}^{-k}\overline{u_0}\hat{s}^k$ tends to ∞, then $\hat{s}^{-k}\underline{v}\hat{s}^k$ also tends to ∞. Thus, the above simple argument does not immediately yield a contradiction.

Instead, we employ a lemma of Raghunathan [8, Lem. 1.7] that provides certain nontrivial relations in Γ. These relations involve elements of both U and V; they provide the crucial tension that leads to a contradiction.

Acknowledgements

The authors would like to thank A.S. Rapinchuk for helpful suggestions. D.W.M. was partially supported by a grant from the National Science Foundation (DMS-0100438).

References

[1] D. Carter, G. Keller, E. Paige, Bounded expressions in $\mathrm{SL}(n, A)$, preprint.
[2] G. Cooke, P.J. Weinberger, On the construction of division chains in algebraic number rings, with applications to SL_2, Commun. Algebra 3 (1975) 481–524.
[3] M.K. Dąbkowski, J.H. Przytycki, A.A. Togha, Non-left-orderable 3-manifold groups, preprint, math.GT/0302098.
[4] É. Ghys, Actions de réseaux sur le cercle, Invent. Math. 137 (1) (1999) 199–231.
[5] É. Ghys, Groups acting on the circle, Ens. Math. 47 (3/4) (2001) 329–407.
[6] B. Liehl, Beschränkte Wortlänge in SL_2, Math. Z. 186 (4) (1984) 509–524.
[7] V.K. Murty, Bounded and finite generation of arithmetic groups, in: K. Dilcher (Ed.), Number Theory, Halifax, NS, 1994, pp. 249–261; CMS Conf. Proc., vol. 15, Amer. Math. Soc., Providence, RI, 1995.
[8] M.S. Raghunathan, On the congruence subgroup problem, II, Invent. Math. 85 (1986) 73–117.
[9] D. Witte, Arithmetic groups of higher $\mathbb{Q}$-rank cannot act on 1-manifolds, P. Am. Math. Soc. 122 (2) (1994) 333–340.