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Rigidity of horospherical foliations

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Abstract. M. Ratner's theorem on the rigidity of horocycle flows is extended to the rigidity of horospherical foliations on bundles over finite-volume locally-symmetric spaces of non-positive sectional curvature, and to other foliations of the same algebraic form.

1. *Introduction*

The geodesic flow on the unit tangent bundle T^1X of a connected, finite-volume manifold X of constant negative curvature is Anosov; the associated strongly stable foliation (or, if you prefer, the strongly unstable foliation) is called the *horospherical foliation* on T^1X. If X is a surface, which means T^1X is 3-dimensional, then the leaves of the horospherical foliation are 1-dimensional; the leaves can be parametrized by arc-length to become the orbits of a flow, called the *horocycle flow*, on T^1X. It was shown by M. Ratner [7] that if the horocycle flows on the unit tangent bundles of two connected, finite-volume surfaces X_1 and X_2 of constant negative curvature are measurably isomorphic, then X_1 and X_2 are isometric (up to the choice of a normalizing constant). In short, Ratner's theorem can be described as saying that horocycle flows are rigid: their measure-theoretic structure completely determines their geometric structure.

THEOREM 1.1. (Ratner Rigidity Theorem [7, Theorem 2]). *Let X_1 and X_2 be two connected, finite-volume surfaces of constant negative curvature, and assume* $\operatorname{vol} X_1 = \operatorname{vol} X_2$. *If $\psi : T^1X_1 \to T^1X_2$ is a measure-preserving, invertible Borel map that conjugates the horocycle flow $H_t^{(1)}$ on T^1X_1 to the horocycle flow $H_t^{(2)}$ on T^1X_2 (i.e., if $\psi \circ H_t^{(2)} = H_t^{(1)} \circ \psi$), then there is an isometry $\phi : X_1 \to X_2$, and some $t_0 \in \boldsymbol{R}$, such that ψ is the differential of ϕ, composed with the translation $H_{t_0}^{(2)}$ (a.e.).* □

If X is a higher-dimensional manifold of constant negative curvature, then the leaves of the horospherical foliation are not 1-dimensional – they are higher-dimensional (immersed) submanifolds of T^1X – so the leaves are not the orbits of a (smooth) flow; but each leaf inherits a Riemannian metric from the metric on T^1X, and the natural analogue in higher dimensions of a conjugacy of horocycle flows is a map that takes each leaf of one horospherical foliation bijectively, via an isometry, onto a leaf of another horospherical foliation. Ratner's theorem extends to this setting.

THEOREM 1.2. (Flaminio [**2**]). *Let X_1 and X_2 be two connected, finite-volume manifolds of constant negative curvature; assume* $\dim X_1 > 2$ *and* $\mathrm{vol}\, X_1 = \mathrm{vol}\, X_2$. *If $\psi: T^1X_1 \to T^1X_2$ is a measure-preserving, invertible Borel map that takes each leaf of the horospherical foliation on T^1X_1 isometrically onto a leaf of the horospherical foliation on T^1X_2, then ψ is the differential of an isometry $\phi: X_1 \to X_2$ (a.e.).* □

The setting of theorem 1.2 can be generalized by considering not the unit tangent bundle, but other bundles over X. For example, the geodesic flow is a factor of the frame flow F_t on the principal bundle $\mathcal{F}X$ of positively-oriented orthonormal frames over X; though F_t is not generally Anosov, it has a strongly stable foliation, which we call the *horospherical foliation* on $\mathcal{F}X$. (Under the factor map $\mathcal{F}X \to T^1X$, each leaf of the horospherical foliation on $\mathcal{F}X$ covers a leaf of the horospherical foliation on T^1X.) It was essentially shown by D. Witte (see Theorem 1.4) that the horospherical foliation on $\mathcal{F}X$ is rigid; L. Flaminio (in conversation) remarked that this suggests the horospherical foliations on intermediate bundles - bundles between $\mathcal{F}X$ and T^1X - should also be rigid. This paper proves the rigidity of the horospherical foliations on these intermediate bundles, and of other similar foliations; the proof is based on M. Ratner's fundamental insights.

Definition. Of course $\mathcal{F}X$ is a principal $\mathrm{SO}(n)$-bundle, where $n = \dim X$. For the purpose of stating Theorem 1.3, we'll say that an $\mathrm{SO}(n)$-bundle $\mathscr{E}$ over X is *intermediate* between $\mathcal{F}X$ and T^1X if there is a pair of surjective $\mathrm{SO}(n)$-bundle maps $\mathcal{F}X \to \mathscr{E}$ and $\mathscr{E} \to T^1X$ whose composition is the natural quotient map $\mathcal{F}X \to T^1X$. (In other words, $\mathscr{E}$ is intermediate between $\mathcal{F}X$ and T^1X if there is some closed subgroup E of $\mathrm{SO}(n-1)$ such that $\mathscr{E}$ is the associated fiber bundle of $\mathcal{F}X$ with fiber $\mathrm{SO}(n)/E$.) The horospherical foliation on $\mathcal{F}X$ pushes to a foliation (called the *horospherical foliation*) on any bundle intermediate between $\mathcal{F}X$ and T^1X.

THEOREM 1.3. *Let X_1 and X_2 be two connected, finite-volume manifolds of constant negative curvature; assume* $\mathrm{vol}\, X_1 = \mathrm{vol}\, X_2$. *Let $\mathscr{E}_i$ be a bundle over X_i intermediate between $\mathcal{F}X_i$ and T^1X_i (for $i = 1, 2$). If there is a measure-preserving, invertible Borel map $\psi: \mathscr{E}_1 \to \mathscr{E}_2$ that takes each leaf of the horospherical foliation on $\mathscr{E}_1$ isometrically onto a leaf of the horospherical foliation on $\mathscr{E}_2$, then X_1 and X_2 are isometric manifolds.* □

The conclusion of Theorem 1.3 is weaker than that of Theorem 1.2: we do not assert that ψ is the differential of an isometry, but only that there is some isometry from X_1 onto X_2; the precise form of ψ (and other aspects of the main theorem) is much easier to state in algebraic, rather than geometric, form: as motivation, we present some highlights of the algebraic formulation of Flaminio's Theorem (1.2) (details are in [**2**]). Let $\tilde{X}$ be the universal cover of a connected, finite-volume manifold X of constant negative curvature. The identity component $G \cong \mathrm{SO}(1, n)$ of the isometry group of $\tilde{X}$ is a simple Lie group; it acts simply transitively on $\mathcal{F}\tilde{X}$

(on the left, say) so, by choosing a basepoint in $\mathcal{F}\tilde{X}$, we may identify $\mathcal{F}\tilde{X}$ with G. There is a (unipotent) subgroup U of G, a so-called *horospherical subgroup*, such that the foliation of G into the orbits of the action of U by right translations is precisely the horospherical foliation on $\mathcal{F}\tilde{X}$. Now X is the quotient of $\tilde{X}$ by a discrete group Γ of isometries; so $\mathcal{F}X = \Gamma\backslash G$, and the horospherical foliation on $\mathcal{F}X$ is the foliation of $\Gamma\backslash G$ into orbits of the action of U by right translations.

Definition. Recall that a matrix A is *unipotent* if it has no eigenvalue other than 1 (i.e., if $A - \mathrm{Id}$ is nilpotent). An element u of a Lie group G is *unipotent* if $\mathrm{Ad}u$ is a unipotent linear transformation on the Lie algebra of G; a subgroup U is *unipotent* if every element of U is a unipotent element of G. Any connected, unipotent subgroup of G is nilpotent (cf. Engel's Theorem [**5**, p. 2]).

There is a compact subgroup M of G that normalizes U and intersects U trivially, such that T^1X is the quotient of $\mathcal{F}X$ by M: $T^1X = \Gamma\backslash G/M$. If sU and tU are two leaves of the horospherical foliation on $\Gamma\backslash G$ whose images sUM and tUM in $\Gamma\backslash G/M$ intersect, then, because M normalizes U, these two images coincide; each leaf in the horospherical foliation on T^1X can be identified (but not in a canonical way) with U. If a leaf in one horospherical foliation is identified with a unipotent group U_1, and a leaf in another horospherical foliation is identified with a unipotent group U_2, then Proposition 2.15 shows the assumption that the restriction of ψ to the leaf U_1 be an isometry onto the leaf U_2 implies the algebraic condition that the restriction of ψ to U_1 be an affine map, i.e., the composition of a group homomorphism and a translation.

With these ideas in mind, let us proceed to the statement of the main Theorem (1.5); we'll need some terminology.

Definition. A discrete subgroup Γ of a Lie group G is a *lattice* if there is a finite G-invariant measure on the homogeneous space $\Gamma\backslash G$; the lattice is *faithful* if Γ contains no nontrivial normal subgroup of G. Any element x of G acts by translation on $\Gamma\backslash G$; namely $T_x : \Gamma_s \mapsto \Gamma sx$ for $s \in G$.

Definition. Let Γ and Λ be closed subgroups of Lie groups G and H, and suppose $\sigma : G \to H$ is a group homomorphism with $\Gamma^\sigma \subset \Lambda$; for any $h \in H$, the map $T_{\sigma,h} : \Gamma\backslash G \to \Lambda\backslash H : \Gamma_s \mapsto \Lambda s^\sigma h$ is said to be *affine*.

THEOREM 1.4. (Witte [**14**, theorem 2.1′]). *Let Γ and Λ be faithful lattices in connected Lie groups G and H. Let T_u and T_v be ergodic, unipotent translations on $\Gamma\backslash G$ and $\Lambda\backslash H$ respectively. If $\psi : \Gamma\backslash G \to \Lambda\backslash H$ is a measure-preserving Borel map that conjugates T_u to T_v, then ψ is an affine map (a.e.).* □

Definition. A connected Lie group G is *reductive* if every connected, solvable, normal subgroup of G is central or, equivalently, if G is locally isomorphic to a direct product $G_1 \times \cdots \times G_n \times A$, where each G_i is simple, and A is abelian [**11**, Theorem 3.16.3, p. 232].

Notation. When we write $V \rtimes M$ for a group, we mean to imply that the group is the *semidirect product* of V and M, i.e., that V and M are closed subgroups of G, that $V \lhd G$, that $VM = G$, and that $V \cap M = e$.

The following definition formalizes the notion that, when leaves of the horospherical foliation are identified with a Lie group, the restriction of ψ to a leaf of the horospherical foliation is an affine map.

Definition. Suppose Lie groups U and $V \rtimes M$ act ergodically on standard Borel spaces $\mathscr{S}$ and $\mathscr{T}$ with finite invariant measures. Assume M is compact, so $\mathscr{T}/M$ is a standard Borel space. A measure-preserving Borel map $\psi: \mathscr{S} \to \mathscr{T}/M$ is *V-affine on each U-orbit* if, for a.e. $s \in \mathscr{S}$, there is a point $t \in \mathscr{T}$, and a surjective, continuous homomorphism $\phi: U \to V$, with $su\psi = tu^{\phi}M$ for all $u \in U$.

MAIN THEOREM 1.5. *Let* Γ *and* Λ *be faithful lattices in connected reductive Lie groups* G *and* H, *and let* U *and* $V \rtimes M$ *be subgroups of* G *and* H. *Assume* U *and* V *are unipotent, and are ergodic on* $\Gamma\backslash G$ and $\Lambda\backslash H$, *respectively; assume* M *is compact and contains no nontrivial normal subgroup of* H; *and assume that* $V \rtimes M$ *is essentially free on* $\Lambda\backslash H$. *If* $\psi: \Gamma\backslash G \to \Lambda\backslash H/M$ *is a measure-preserving Borel map that is V-affine on each U-orbit, then* ψ *lifts to an affine map* $\psi': \Gamma\backslash G \to \Lambda\backslash H$. *I.e., there is a measure-preserving affine map* $\psi': \Gamma\backslash G \to \Lambda\backslash H$ *with* $\Gamma s\psi = \Lambda s\psi' M$ *(a.e.).*

There are two parts to the proof of the main Theorem. First, an abstract argument (Theorem 3.1) shows M can be replaced by $C_M(V)$; this is a big gain because V acts by translations on $\Lambda\backslash H/C_M(V)$, so we now have a group action, instead of a mere foliation. If it happens to be the case that no compact subgroup of H centralizes V, then we have reduced to the case where $M = e$; Theorem 1.4 applies and we are done. In general, however, we need to generalize Theorem 1.4; this is the second part of the proof (Theorem 4.1).

Application 1. Let $X = G/K$ be a finite-volume locally-symmetric space of non-positive sectional curvature; assume, for simplicity, that no flat subspace is locally a direct factor of X (so G is semisimple). The horospherical foliation on $\mathscr{F}X$ or on T^1X will often not be ergodic; almost every ergodic component of the horospherical foliation is a sub-bundle of $\mathscr{F}X$ or T^1X of the form $\Gamma\backslash G/M$, for some subgroup M of K. The main theorem implies that the restriction of the horospherical foliation to these ergodic components is rigid.

Theorem 4.1 settles the isomorphism question for a natural class of actions of semisimple Lie groups.

Application 2. Suppose G, H_1, H_2 are connected, noncompact, semisimple Lie groups with finite center, and let Λ_i be a faithful lattice in H_i that projects density into the maximal compact factor of H_i. Embed G in H_1 and H_2, and assume G acts ergodically on $\Lambda_i\backslash H_i$. Let M_i be a compact subgroup of H_i that centralizes G, and contains no nontrivial normal subgroup of H_i; then any measure-theoretic isomorphism from the action of G by translations on $\Lambda_1\backslash H_1/M_1$ to the action of G by translations on $\Lambda_2\backslash H_2/M_2$ lifts to an affine map $\Lambda_1\backslash H_1 \to \Lambda_2\backslash H_2$ (a.e.).

Let G, H_i, Λ_i, and M_i be as in Application 1. As one step in an interesting argument (in preparation) on cocycles of an action of a semisimple group, R. J. Zimmer wanted to know that if H_1 and H_2 are entirely different groups, then the G-action on $\Lambda_2\backslash H_2/M_2$ cannot be a factor of the G-action on $\Lambda_1\backslash H_1/M_1$, or even of a finite extension thereof. A technical version (Theorem 4.1′) of Theorem 4.1 proves this.

Application 3. Suppose G, H_1, H_2 are connected, noncompact, semisimple Lie groups with finite center, and let Λ_i be a faithful lattice in H_i that projects densely into the maximal compact factor of H_i. Embed G in H_1 and H_2, and assume G acts ergodically on $\Lambda_i \backslash H_i$. Let M_i be a compact subgroup of H_i that centralizes G, and contains no nontrivial normal subgroup of H_i. If the G-action on $\Lambda_2 \backslash H_2 / M_2$ is a factor of some finite extension of the G-action on $\Lambda_1 \backslash H_1 / M_1$, then H_2 is locally isomorphic to a factor group of H_1.

Remark. From the geometric point of view, it is natural to ask whether horospherical foliations on the unit tangent bundles of manifolds of *nonconstant* negative curvature are rigid; even for surfaces, this is not known. (J. Feldman and D. Ornstein [**1**] have proved a result of this type for surfaces, but they do not parametrize the leaves of the horocycle foliation by arc-length.)

In the main theorem, the assumption on the restriction of ψ to leaves of the foliation is necessary. For example, M. Ratner [**6**, Theorem 3] showed that the horocycle foliation on the unit tangent bundle of any connected, finite-volume surface of constant negative curvature is measurably equivalent, via a map that is a homeomorphism on leaves, to that on any other.

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2. *Preliminaries*

Our terminology follows Zimmer [**15**].

2.1. *Ergodic theory*

Definition. Suppose a Lie group Y acts on a Borel space $\mathcal{T}$ with quasi-invariant measure. The action is *free* if, whenever $ty = t$ with $t \in \mathcal{T}$ and $y \in Y$, then $y = e$; the action is *essentially free* if there is a conull Y-invariant subset $\mathcal{T}'$ of $\mathcal{T}$ such that the restricted action of Y on $\mathcal{T}'$ is free.

Definition. Suppose Lie groups U and V act on standard Borel spaces $\mathcal{S}$ and $\mathcal{T}$, respectively; let M be a compact group acting on $\mathcal{T}$. We say a Borel map $\psi : \mathcal{S} \to \mathcal{T}/M$ is *affine* for U (via V) if, for each $u \in U$, there is some $\tilde{u} \in V \cap N_H(M)$ such that ψ conjugates the action of u on $\mathcal{S}$ to the action of $\tilde{u}$ on $\mathcal{T}/M$, i.e., if $su\psi = s\psi \cdot \tilde{u}$ for a.e. $s \in \mathcal{S}$.

Definition. If $\mathcal{S}'$ is a conull subset of a Borel measure space $(\mathcal{S}, \mu)$, then we say μ is *supported* on $\mathcal{S}'$ - even if $\mathcal{S}'$ is not a closed set.

Definition. Given Borel spaces $\mathcal{S}$ and $\mathcal{T}$, and a Borel map $\psi : \mathcal{S} \to \mathcal{T}$, any probability measure μ on $\mathcal{S}$ *pushes* to a probability measure $\psi_* \mu$ on $\mathcal{T}$ given by $\int_{\mathcal{T}} f d(\psi_* \mu) = \int_{\mathcal{S}} \psi \circ f \, d\mu$.

Definition. Given probability measures μ_1 and μ_2 on Borel spaces $\mathscr{S}_1$ and $\mathscr{S}_2$ respectively, a probability measure μ on $\mathscr{S} \times \mathscr{T}$ is a *joining* of μ_1 and μ_2 if, under the projection $\mathscr{S}_1 \times \mathscr{S}_2 \to \mathscr{S}_i$, the measure μ pushes to μ_i (for $i = 1, 2$).

Definition. Given a joining μ of $(\mathscr{S}_1, \mu_1)$ and $(\mathscr{S}_2, \mu_2)$, there is (see [**3**, Theorem 5.8, p. 108]) an essentially unique family $\{\mu_s : s \in \mathscr{S}_1\}$ of measures on $\mathscr{S}_2$ such that, for any measurable $A \subset \mathscr{S}_1 \times \mathscr{S}_2$, $\mu(A) = \int_{\mathscr{S}_1} \mu_s(A \cap (\{s\} \times \mathscr{S}_2))\, d\mu_1(s)$; these measures μ_s are the *fibers of* μ *over* $\mathscr{S}_1$. We say μ has *finite fibers over* $\mathscr{S}_1$ if the support of a.e. fiber is a finite set.

Remark. If $\psi : (\mathscr{S}_1, \mu_1) \to (\mathscr{S}_2, \mu_2)$ is a measure-preserving Borel map, then the graph of ψ supports a joining of $(\mathscr{S}_1, \mu_1)$ and $(\mathscr{S}_2, \mu_2)$, of which each fiber over $\mathscr{S}_1$ is supported on a single point.

Definition. Let T_1 and T_2 be measure-preserving maps on Borel probability spaces $(\mathscr{S}_1, \mu_1)$ and $(\mathscr{S}_2, \mu_2)$. A measure μ on $\mathscr{S}_1 \times \mathscr{S}_2$ is a *joining* of $(T_1, \mathscr{S}_1, \mu_1)$ and $(T_2, \mathscr{S}_2, \mu_2)$ if (1) μ is a joining of μ_1 and μ_2; and (2) μ is $T_1 \times T_2$-invariant.

More generally, suppose Lie groups U_1 and U_2 act, with invariant probability measures μ_1 and μ_2, on Borel spaces $\mathscr{S}_1$ and $\mathscr{S}_2$, respectively. Given a continuous homomorphism $\phi : U_1 \to U_2$, a probability measure μ on $\mathscr{S}_1 \times \mathscr{S}_2$ is a *joining of* $(U_1, \mathscr{S}_1)$ *and* $(U_2, \mathscr{S}_2)$ *under* ϕ if (1) μ is a joining of μ_1 and μ_2; and (2) for each $u \in U_1$, the measure μ is (u, u^{ϕ})-invariant (where $U_1 \times U_2$ acts on $\mathscr{S}_1 \times \mathscr{S}_2$ by $(s_1, s_2) \cdot (u_1, u_2) = (s_1 u_1, s_2 u_2)$).

LEMMA 2.1. *Suppose $\mathscr{S}$ and $\mathscr{T}$ are standard probability spaces, and a compact group M acts on $\mathscr{T}$, preserving the measure. Given a measure-preserving Borel map $\psi : \mathscr{S} \to \mathscr{T}/M$, let* GRAPH $= \{(s, t) \in \mathscr{S} \times \mathscr{T} \mid s\psi = tM\}$. *Then* GRAPH *is a Borel subset that supports a (unique) M-invariant joining of the measures on $\mathscr{S}$ and $\mathscr{T}$.*

Proof. Since ψ is measure-preserving, its graph supports a joining μ' of the measures on $\mathscr{S}$ and $\mathscr{T}/M$. This joining has a natural M-invariant lift to a measure μ on $\mathscr{S} \times \mathscr{T}$; namely, $\int f\, d\mu = \int \int_M f(sk)\, dk\, d\mu'(s)$. It is clear that this lift μ is supported on the inverse image, under the quotient map $\mathscr{S} \times \mathscr{T} \to \mathscr{S} \times \mathscr{T}/M$, of the graph of ψ. This inverse image - a Borel set - is GRAPH. ∎

LEMMA 2.2. *Suppose T_1 and T_2 are ergodic measure-preserving maps on Borel probability spaces $(\mathscr{S}_1, \mu_1)$ and $(\mathscr{S}_2, \mu_2)$, and μ is a joining of $(T_1, \mathscr{S}_1, \mu_1)$ and $(T_2, \mathscr{S}_2, \mu_2)$. Then almost every ergodic component of $(T_1 \times T_2, \mathscr{S}_1 \times \mathscr{S}_2)$ is a joining of $(T_1, \mathscr{S}_1, \mu_1)$ and $(T_2, \mathscr{S}_2, \mu_2)$.*

Proof. Because μ is the integral of its ergodic components, and μ pushes to the measure μ_1 on $\mathscr{S}_1$, it follows that μ_1 is the integral of the measures obtained by pushing the ergodic components of μ to $\mathscr{S}_1$; since μ_1 is ergodic, this implies almost every ergodic component of μ pushes to the measure μ_1 on $\mathscr{S}_1$. Similarly, almost every ergodic component of μ pushes to the measure μ_2 on $\mathscr{S}_2$. □

2.1. *Lie theory*

All Lie groups are assumed to be second countable.

Notation. For subgroups X and Y of a Lie group G, we use $C_X(Y)$ and $N_X(Y)$ to denote the centralizer and normalizer of Y in X, respectively; X^0 is the identity component of X.

LEMMA 2.3. *If G is reductive connected Lie group, then $G = Z(G) \cdot [G, G]$, and $Z(G) \cap [G, G]$ is discrete.* ■

LEMMA 2.4. (cf. [**14**, Proposition 2.6]). *Let Γ be a faithful lattice in a connected, reductive Lie group G, and assume there is an ergodic unipotent translation on $\Gamma\backslash G$. Then $Z(G)$ is compact, and Γ projects densely into the maximal compact semisimple factor of G.* □

LEMMA 2.5. *Let Γ be a lattice in a connected, reductive Lie group G, and assume $Z(G)$ is compact. Suppose ψ is a measurable function on $\Gamma\backslash G$, and let*

$$X = \{g \in G \mid sg\psi = s\psi \text{ for a.e. } s \in \Gamma\backslash G\}.$$

Then there is a closed normal subgroup N of G, contained in X, such that X/N is compact.

Proof. Because G is reductive and $Z(G)$ is compact, one can show the center of any quotient group of G is compact. The Mautner phenomenon [**4**, Theorem 1.1] implies there is a closed normal subgroup N of G, contained in X, such that X projects to an Ad-precompact subgroup of G/N; since X is closed and $Z(G/N)$ is compact, this implies X/N is compact. □

LEMMA 2.6. *Let G be a connected, reductive Lie group whose center is compact; let U be the identity component of a maximal unipotent subgroup of G, and let K be the maximal compact semisimple factor of G. If M is a compact subgroup of G normalized by both U and K, then $M \lhd G$.*

Proof. Suppose first that $Z(G) = e$; so G is a real algebraic group. Since M is compact, it is a reductive real algebraic subgroup of G, so there is a Cartan involution $(*)$ of G that normalizes M [**5**, § 2.6, p. 11]; hence M is normalized by $\langle U, U^*, K\rangle = G$.

Now, even if $Z(G) \neq e$, the preceding paragraph shows $\mathrm{Ad}M \lhd \mathrm{Ad}G$, so $M \cdot Z(G) \lhd G$. Since $M \cdot Z(G)$ is compact, then Lemma 2.10 implies every unipotent element of G centralizes $M \cdot Z(G)$; in particular, every unipotent element of G normalizes M. These unipotent elements, together with $Z(G)$ and the maximal compact semisimple factor K, generate G, so $M \lhd G$ as desired. ■

LEMMA 2.7. *Let C, M, and N be subgroups of an abstract group G, so that CM is a subgroup of finite index in G. If $M \subset N$, then $(C \cap N)M$ is a subgroup of finite index in N. In fact, $|N: (C \cap N)M| \leq |G: CM|$.* □

LEMMA 2.8. *If M is a compact subgroup of a connected Lie group H, then $C_H(M) \cdot M$ is of finite index in $N_H(M)$.*

Proof. The Ado-Iwasawa Theorem [**5**, § P.1.4, p. 3] asserts there is a locally faithful finite-dimensional representation $\pi: H \to G = \mathrm{GL}_n(\boldsymbol{R})$. Let $C = C_G(M^\pi)^{\pi^{-1}}$ and $N = N_G(M^\pi)^{\pi^{-1}}$. It suffices to show $|N: C_H(M) \cdot M| < \infty$.

Step 1. $|C: C_H(M)| < \infty$. Now M/M^0 is finite, so there is a finite subset $\{k_1, \ldots, k_n\}$ of M with $M = \langle M^0, k_1, \ldots, k_n\rangle$. Let F be the unique maximal finite subgroup of $\ker \pi$, namely, the intersection of $\ker \pi$ with any maximal compact subgroup of H. Then C centralizes MF/F, so for any $k \in M$, we have $|k^C| \leq |F| < \infty$. So a subgroup of finite index in C centralizes k; so a subgroup C_0 of finite index centralizes all of $k_1, \ldots, k_n$. But, being contained in C, the subgroup C_0 must also centralize M/F; therefore, C_0 centralizes M^0. Combining these two conclusions yields $C_0 \subset C_H(M)$.

Step 2. $|N: C \cdot M| < \infty$. Since $\ker \pi \subset C$, it suffices to show $|N^\pi: C^\pi \cdot M^\pi| < \infty$. For this, it suffices to show $|N_G(M^\pi): C_G(M^\pi) \cdot M^\pi| < \infty$ (see Lemma 2.7); so we may assume $H = G$ and π is the identity map. Since M is compact, it is (the real points of) a real algebraic subgroup of G [**15**, p. 40]. So $N_G(M)$ is also an algebraic subgroup; therefore, $|N_G(M): N_G(M)^0| < \infty$ [**5**, § P.2.4, p. 10]. Let $\mathcal{N}$, $\mathcal{C}$, and $\mathcal{M}$ be the Lie algebras of $N_G(M)$, $C_G(M)$ and M. Now any representation of a compact group is completely reducible, so there is a M-invariant complement V to $\mathcal{M}$ in $\mathcal{N}$. But M centralizes $\mathcal{N}/\mathcal{M}$, so it follows that $V \subset \mathcal{C}$; therefore, $\mathcal{N} = \mathcal{C} + \mathcal{M}$; therefore, $N_G(M)^0 \subset C_G(M)^0 \cdot M^0$. □

LEMMA 2.9. *Suppose u_i is a unipotent element of a Lie group G_i, and M_i is a compact, normal subgroup of $C_i = C_{G_i}(u_i)^0$ (for $i = 1, 2$), and suppose $\sigma: C_1/M_1 \to C_2/M_2$ is an isomorphism. If v is a unipotent element of C_1/M_1, then there is a unipotent element v' of G_2, contained in C_2, with $v^\sigma = v'M_2$.*

Proof. Lemma 2.8 implies $C_2 = C_{C_2}(M_2) \cdot M_2$, so there is some $v' \in C_{C_2}(M_2)$ with $v^\sigma = v'M_2$; we claim v' is unipotent. Since v is a unipotent element of C_1/M_1 and σ is an isomorphism, v' is unipotent on C_2/M_2; since v' centralizes M_2, this implies v' is a unipotent element of C_2. It is an elementary fact from linear algebra that if T and N are commuting endomorphisms of a finite-dimensional vector space, such that both N and the restriction of T to $\ker(N - \mathrm{Id})$ are unipotent, then T is unipotent; applying this with $T = \mathrm{Ad}\, v'$ and $N = \mathrm{Ad}\, u_2$, we conclude that v' is a unipotent element of G_2. ■

LEMMA 2.10. *Let u be a unipotent element, and M be a compact subgroup, of a connected, reductive Lie group G. If u normalizes M, then u centralizes M.*

Proof. Being a unipotent element of the centerless, semisimple real algebraic group $\mathrm{Ad}\,G$, the element $v = \mathrm{Ad}\,u$ belongs to a one-parameter unipotent subgroup v^r of $\mathrm{Ad}\,G$ (cf. [**5**, § 2.4, p. 10]); lift v^r to a one-parameter subgroup $\hat{v}^r$ of G. For any $k \in M$, the fact that u normalizes M implies that the image of the map $\boldsymbol{R} \to \mathrm{Ad}\,G: r \mapsto \mathrm{Ad}\,k^{\hat{v}^r}$ is contained in $\mathrm{Ad}\,M$; this means the map is bounded. But, because u is unipotent, the map is a polynomial; a bounded polynomial is constant, so $\mathrm{Ad}\,k^{\hat{v}^r} = \mathrm{Ad}\,k$, for all $r \in \boldsymbol{R}$. Hence $k^{\hat{v}^r} = k \pmod{Z(G)}$, for all $r \in \boldsymbol{R}$. Since $Z(G) \cap [G, G]$ is discrete, and $\boldsymbol{R}$ is connected, this implies $\hat{v}$ centralizes k; since $k \in \hat{M}$ was arbitrary, this means $\hat{v}$ centralizes M; it follows that u centralizes M, as desired. □

LEMMA 2.11. *Let $\underline{u}$ be an ad-nilpotent element of a (real or complex) reductive Lie algebra $\mathcal{G}$. If, for some $\underline{g} \in \mathcal{G}$, the commutator $[\underline{u}, \underline{g}]$ is ad-semisimple and centralizes $\underline{u}$, then $[\underline{u}, \underline{g}] = 0$.*

Proof. Let $\underline{k}=[\underline{u},\underline{g}]$. Then

$$0=[\underline{k},\underline{k}]=[\underline{k},[\underline{u},\underline{g}]]=[[\underline{k},\underline{u}],\underline{g}]+[\underline{u},[\underline{k},\underline{g}]]=[0,\underline{g}]+[\underline{u},[\underline{k},\underline{g}]],$$

so $[\underline{k},\underline{g}]\in C_{\mathscr{G}}(\underline{u})$. Since $\underline{k}$ is ad-semisimple, then $\underline{g}\in C_{\mathscr{G}}(\underline{u})+C_{\mathscr{G}}(\underline{k})$, so we may assume $\underline{g}\in C_{\mathscr{G}}(\underline{k})=\mathscr{C}$; hence, both $\underline{g}$ and $\underline{u}$ are in $\mathscr{C}$, so $\underline{k}=[\underline{u},\underline{g}]\in[\mathscr{C},\mathscr{C}]$. But $\mathscr{C}$ is reductive, which implies $[\mathscr{C},\mathscr{C}]\cap Z(\mathscr{C})=0$, so this implies $\underline{k}=0$. ■

THEOREM 2.12. ('the Ratner Property', cf. [**13**, Theorem 6.1]). *Let u be a unipotent element of a Lie group G, and let M be a compact subgroup of G that centralizes u. Given any neighborhood Q of e in $C_G(u)$, there is a compact subset ∂Q of Q, disjoint from M, such that, for any $\varepsilon>0$ and $M>0$, there are α, $\delta>0$ such that,*

if $s,t\in\Gamma\backslash G$ with $d(s,t)<\delta$, then either:

(a) *$s=tc$ for some $c\in C_G(u)$ with $d(e,c)<\delta$, or*

(b) *there are $N>0$ and $q\in\partial Q$ such that $d(su^n,tu^nq)<\varepsilon$ whenever $N\le n\le N+\max(M,\alpha N)$.*

Proof. This is precisely the statement of the Ratner property as it appears in [**13**, Theorem 6.1], except that we need ∂Q to be disjoint from M, rather than just $e\notin\partial Q$; only minor changes in the proof are needed. The key observation is that, in [**13**, Lemma 6.2], π need not be a projection onto the kernel of T; namely, π may be any projection onto the intersection of the kernel of T with the image of T. Therefore, in [**13**, Proposition 6.3], q may be chosen in the intersection of ker T with the image of T; this shows that the subset ∂Q in the Ratner property may be chosen to be the exponential of a small set (not containing 0) in $\mathscr{H}=[\mathscr{G},\underline{u}]\cap C_{\mathscr{G}}(\underline{u})$, where $\exp(\underline{u})=u$. But Lemma 2.11 implies that $\mathscr{M}$ does not intersect $\mathscr{H}$, so $\partial Q\cap M=$ as desired. □

COROLLARY 2.13. ('The Relativized Ratner Property'). *Let u be a unipotent element of a Lie group G, and let M be a compact, normal subgroup of $C_G(u)^0$. Given any neighborhood Q of e in $C_G(u)/M$, there is a compact subset ∂Q of $Q-e$ such that, for any $\varepsilon>0$ and $M>0$, there are α, $\delta>0$ such that,*

if $s,t\in\Gamma\backslash G/M$ with $d(s,t)<\delta$, then either:

(a) *$s=tc$ for some $c\in C_G(u)$ with $d(e,c)<\delta$, or*

(b) *there are $N>0$ and $q\in\partial Q$ such that $d(su^n,tu^nq)<\varepsilon$ whenever $N\le n\le N+\max(M,\alpha N)$.* ■

2.3. *Isometries and affine maps*

In the geometric formulation of Flaminio's Theorem (1.2), it is assumed that the restriction of ψ to each leaf of the horospherical foliation is an isometry; Proposition 2.15 shows that the natural algebraic formulation would assume that the restriction is an affine map.

LEMMA 2.14. *Let U and V be Lie groups, and $\psi:U\to V$ be any map. Then ψ is an affine map iff it conjugates the group of (left) translations on U into the group of (left) translations on V, i.e., iff, for each $u\in U$, there is some $v\in V$, such that $T_u\circ\psi=\psi\circ T_v$ (where T_u or T_v is the (left) translation by u on U or by v on V).* ■

PROPOSITION 2.15. (E. N. Wilson). *Let U and V be connected nilpotent Lie groups; make U and V Riemannian manifolds by supplying each of them with a left-invariant metric; then every isometry σ of U onto V is an affine map.*

Proof. Obviously, the nilpotent group U acts simply transitively on itself by left translations, and these translations are isometries; so σ conjugates this nilpotent group of left translations on U to a nilpotent group of isometries of V acting simply transitively on V. A theorem of E. N. Wilson [**12**, Theorem 2(4)] asserts that the group of left translations on V is the unique nilpotent group of isometries of V acting simply transitively on V, so we conclude that σ conjugates the left translations on U to the left translations on V. Hence Lemma 2.14 asserts that σ is an affine map. □

3. *From a foliation to an action*

THEOREM 3.1. *Let Lie groups U and $V \rtimes M$ act on standard Borel probability spaces $(\mathcal{S}, \sigma)$ and $(\mathcal{T}, \tau)$, respectively. Assume the actions of U and V are measure-preserving and ergodic, that M is compact, and that $V \rtimes M$ is essentially free on $\mathcal{T}$. If a measure-preserving Borel map $\psi: \mathcal{S} \to \mathcal{T}/M$ is V-affine on each U-orbit, then ψ lifts to a map $\mathcal{S} \to \mathcal{T}/C_M(V)$ that is affine for U via V.*

Proof. Let GRAPH $= \{(s, t) \in \mathcal{S} \times \mathcal{T} \mid s\psi = tM\}$; lemma 2.1 asserts GRAPH is a Borel set that supports a (unique) M-invariant joining μ of $(\mathcal{S}, \sigma)$ and $(\mathcal{T}, \tau)$.

Step 1. There is a conull U-invariant subset $\mathcal{S}'$ of $\mathcal{S}$ such that, for all $s \in \mathcal{S}'$ and all $t \in \mathcal{T}$ with $s\psi = tM$, there is a homomorphism $\phi = \phi_{s,t}: U \to V$ with $su\psi = tu^{\phi}M$ for all $u \in U$. Because ψ is V-affine on each U-orbit, there is a conull Borel subset A of $\mathcal{S}$ such that, for any $s \in A$, there is some $t \in \mathcal{T}$ and a surjective homomorphism $\phi: U \to V$ with $su\psi = tu^{\phi}M$ for all $u \in U$. Just because A is conull, there is a conull subset B of A such that the conull U-invariant subset BU of $\mathcal{S}$ is Borel [**15**, Lemma B.8(i), pp. 199-200].

We can verify as follows that $\mathcal{S}' = BU$ is as described. Given $s = bu_0 \in BU = \mathcal{S}'$, there is some $t_0 \in \mathcal{T}$, and a surjective homomorphism $\phi_0: U \to V$, such that $bu\psi = t_0u^{\phi_0}M$ for all $u \in U$. For any $t \in \mathcal{T}$ with $s\psi = tM$, we have $t_0u_0^{\phi_0}M = bu_0\psi = s\psi = tM$, so there is some $k \in M$ with $t_0u_0^{\phi_0}k = t$. The surjective homomorphism ϕ: $U \to V$: $u \mapsto k^{-1}u^{\phi_0}k$ satisfies

$$su\psi = bu_0u\psi = t_0(u_0u)^{\phi_0}M = (t_0u_0^{\phi_0}k)k^{-1}u^{\phi_0}M = tu^{\phi}M,$$

as desired.

Remark. There is, of course, no loss in assuming $\mathcal{S}' = \mathcal{S}$. We may also assume $V \rtimes M$ acts freely on $\mathcal{T}$, by removing a null set from $\mathcal{T}$, and removing the inverse image of this null set from $\mathcal{S}$.

Definition. Let Hom (U, V) be the set of all continuous homomorphisms $U \to V$, and give Hom (U, V) the countably-generated Borel structure generated by the basic sets $\mathcal{B}_{u,A} = \{\phi \in \text{Hom}(U, V) \mid u^{\phi} \in A\}$, where u ranges over a countable dense subset of U, and A ranges over a countable collection of Borel sets generating the Borel structure on V.

Step 2. *There is a Borel map* ϕ : GRAPH $\to$ Hom (U, V) : $(s, t) \mapsto \phi_{s,t}$ *such that, for all* $(s, t) \in$ GRAPH *and all* $u \in U$, *we have* $su\psi = tu^{\phi_{s,t}}M$. For any $(s, t) \in$ GRAPH, Step 1 (amplified by the subsequent remark) asserts there is a homomorphism $\phi_{s,t}$. Since $V \rtimes M$ acts freely on $\mathcal{T}$, this homomorphism is uniquely determined by (s, t). So there is a map GRAPH $\to$ Hom (U, V), but perhaps it is not obvious that the map is measurable.

To show the measurability of ϕ, let u be any element of U, and let A be any Borel subset of V; consider the Borel set

$$\text{GRAPH}_{u,A} = \{(s, t, v) \in \mathcal{S} \times \mathcal{T} \times A \mid s\psi = tM \text{ and } su\psi = tvM\}.$$

Notice that, for any $(s, t) \in$ GRAPH, we have

$$\phi_{s,t} \in \mathcal{B}_{u,A} \Leftrightarrow u^{\phi_{s,t}} \in A \Leftrightarrow \exists_{v \in A}(s, t, v) \in \text{GRAPH}_{u,A};$$

thus $(B_{u,A})\phi^{-1}$ is the image of the natural projection π: GRAPH$_{u,A} \to$ GRAPH. Since π is injective (each fiber is, at most, the single point $(s, t, u^{\phi_{s,t}})$), the image of π is Borel [**15**, Theorem A.4, p. 195]; thus the inverse image, under ϕ, of each basic Borel set in Hom (U, V) is a Borel subset of GRAPH, so ϕ is Borel measurable.

Definition. Let U act on GRAPH by $(s, t) \cdot u = (su, tu^{\phi_{s,t}})$; it is easy to see this action is measure-preserving, because it commutes with the action of M. Let (Ω, ω) be (almost) any ergodic component of (U, GRAPH, μ).

Step 3. *The measure* ω *pushes to the measure* σ *on* $\mathcal{S}$; $\phi_{s,t} = \phi \in$ Hom (U, V) *is (essentially) constant on* (Ω, ω); *and each fiber of* ω *over* $\mathcal{S}$ *is supported on a single* $C_M(V)$*-orbit in* $\mathcal{T}$ *(a.e.)*. (1) The proof of Lemma 2.2 shows that almost every ergodic component, such as ω, of (X, GRAPH, μ) pushes to σ on S. (2) A routine calculation shows $(s, t) \mapsto \phi_{s,t}$ is U-invariant, so it is essentially constant on almost any ergodic component: there is an ω-conull subset Ω' of Ω on which $\phi_{s,t}$ is constant. (3) For almost every $s \in \mathcal{S}$, the fiber ω_s of ω over s is supported on $\Omega' \cap (\{s\} \times \mathcal{T})$, because $\omega(\Omega') = 1$. For $(s, t) \in$ GRAPH, $k \in M$, and $u \in U$, a routine calculation shows $u^{\phi_{s,tk}} = k^{-1}u^{\phi_{s,t}}k$; so, if $\phi_{s,tk} = \phi_{s,t}$, then $k \in C_M(V)$; so, for $s \in \mathcal{S}$ and $t, t' \in \mathcal{T}$, if both (s, t) and (s, t') belong to Ω', then t and t' belong to the same $C_M(V)$-orbit on V; i.e., $\Omega' \cap (\{s\} \times \mathcal{T})$ is a subset of a single $C_M(V)$-orbit on $\mathcal{T}$.

Step 4. ω *is a joining of* $(U, \mathcal{S}, \sigma)$ *and* $(V, \mathcal{T}, \tau)$ *via* ϕ. By definition, any ergodic component, such as ω, of the U-action on GRAPH must be (u, u^{ϕ})-invariant, and Step 3 showed ω pushes to σ on $\mathcal{S}$, so we need only show ω pushes to τ on $\mathcal{T}$. (This is not quite trivial, because V does not act on (but only foliates) $\mathcal{T}/M$.) Because ω is (u, u^{ϕ})-invariant for all $u \in U$, and $U^{\phi} = V$, the measure τ' to which ω pushes on $\mathcal{T}$ must be V-invariant; and, since M normalizes V, any M-translate of τ' is also V-invariant. The M-action on GRAPH commutes with the U-action, so every M-translate of ω is U-invariant; because μ is M-invariant, this implies every M-translate of ω is an ergodic component of μ; because the support of each fiber of μ over $\mathcal{S}$ is a single M-orbit, this implies μ is the average of all the M-translates of ω. Pushing to $\mathcal{T}$, we conclude that τ is the average of all the M-translates of τ'; since τ is ergodic for V, and each of these M-translates is V-invariant, this implies $\tau' = \tau$.

Step 5. *ψ lifts to a map $\psi_0: \mathcal{S} \to \mathcal{T}/C_M(V)$ that is affine for U via V.* Let $M' = C_M(V)$. It follows from Step 4 that, under the quotient map $\mathcal{S} \times \mathcal{T} \to \mathcal{S} \times \mathcal{T}/M'$, the ergodic component ω pushes to a joining ω' of $(U, \mathcal{S}, \sigma)$ and $(V, \mathcal{T}/M', \tau)$ via ϕ. Since (by Step 3) each fiber of ω over $\mathcal{S}$ is supported on a single M'-orbit, each fiber of ω' over $\mathcal{S}$ is supported on a single point; so ω' is the joining associated to some measure-preserving Borel map $\psi_0: \mathcal{S} \to \mathcal{T}/M'$. Because ω' is a joining of $(U, \mathcal{S})$ and $(V, \mathcal{T}/M)$, the map ψ_0 is affine for U via V. Because ω is supported on GRAPH, the map ψ_0 is a lift of ψ. □

4. *Rigidity of translations*

THEOREM 4.1. *Let Γ and Λ be faithful lattices in connected, reductive Lie groups G and H, and let M be a compact subgroup of H that contains no nontrivial normal subgroup of H. Suppose u and $\tilde{u}$ are ergodic, unipotent translations on $\Gamma\backslash G$ and $\Lambda\backslash H$, and assume $\tilde{u} \in N_H(M)$. If $\psi: \Gamma\backslash G \to \Lambda\backslash H/M$ is a measure-preserving Borel map that conjugates the translation by u on $\Gamma\backslash G$ to the translation by $\tilde{u}$ on $\Lambda\backslash H/M$, then ψ lifts to an affine map $\psi': \Gamma\backslash G \to \Lambda\backslash H$ (a.e.).*

We prove Theorem 4.1 by reducing to a known special case: Theorem 1.4 settles the case where $M = e$. Several of the arguments to be used in the reduction were used in proving the special case; where practical, we refer the reader to the relevant parts of [**13**] instead of repeating the arguments here. The work is based on fundamental ideas developed by M. Ratner [**7**, **8**, **9**]; a short exposition of some of these ideas appears in [**14**, § 2]; a survey of Ratner's work appears in [**10**].

For technical reasons (discussed after Step 4 of the proof), measure-preserving maps are not general enough: ψ should be allowed to be a joining with finite fibers over $\Gamma\backslash G$, so we will prove Theorem 4.1′ instead of Theorem 4.1. For our purposes, a technique developed by M. Ratner (see [**8**, Lemmas 4.2 and 4.4] and [**9**]) allows us to treat finite-fiber joinings in essentially the same way as maps, but at the cost of severe notational complications; I will usually pretend that ψ is a map, and leave it to the reader to transfer the proof to finite-fiber joinings.

THEOREM 4.1′. *Let Γ and Λ be faithful lattices in connected, reductive Lie groups G and H, and let M be a compact subgroup of H that contains no nontrivial normal subgroup of H. Suppose u and $\tilde{u}$ are ergodic, unipotent translations on $\Gamma\backslash G$ and $\Lambda\backslash H$, and assume $\tilde{u} \in N_H(M)$. If ψ is an ergodic joining of the translation by u on $\Gamma\backslash G$ and the translation by $\tilde{u}$ on $\Lambda\backslash H/M$, and if ψ has finite fibers over $\Gamma\backslash G$, then ψ is an affine joining; i.e., there is a finite cover G' of G, a lattice Γ' in G', and a measure-preserving affine map $\phi: \Gamma'\backslash G' \to \Lambda\backslash H$ such that, under the natural map $\Gamma'\backslash G' \times \Lambda\backslash H \to \Gamma\backslash G \times \Lambda\backslash H/M$, the joining on $\Gamma'\backslash G' \times \Lambda\backslash H$ associated to ϕ pushes to ψ.*

The proof of Theorem 4.1′ reduces it not quite to Theorem 1.4, but to the following more general version that allows finite-fiber joinings.

THEOREM 1.4′. [**14**, Theorem 2.1]. *Let Γ and Λ be faithful lattices in connected Lie groups G and H. Let u and $\tilde{u}$ be ergodic, unipotent translations on $\Gamma\backslash G$ and $\Lambda\backslash H$. If ψ is an ergodic joining of the translation by u on $\Gamma\backslash G$ and the translation by $\tilde{u}$ on*

$\Lambda\backslash H$, and if ψ has finite fibers over $\Gamma\backslash G$, then ψ is an affine joining; i.e., there is a finite cover G' of G, a lattice Γ' in G', and a measure-preserving affine map $\phi: \Gamma'\backslash G' \to \Lambda\backslash H$ such that, under the natural map $\Gamma'\backslash G' \times \Lambda\backslash H \to \Gamma\backslash G \times \Lambda\backslash H$, the joining on $\Gamma'\backslash G' \times \Lambda\backslash H$ associated to ϕ pushes to ψ. □

Proof (of Theorem 4.1/4.1'). By the descending chain condition on compact subgroups of H, we may assume there is no closed, proper subgroup M' of M, normalized by $\tilde{u}$, such that ψ lifts to a measure-preserving Borel map $\Gamma\backslash G \to \Lambda\backslash H/M'$ that conjugates the translation by u on $\Gamma\backslash G$ to the translation by $\tilde{u}$ on $\Lambda\backslash H/M'$. Let

$$\mathrm{GRAPH} = \{(s, t) \in \Gamma\backslash G \times \Lambda\backslash H \mid s\psi = tM\};$$

as explained Lemma 2.1, GRAPH supports a unique M-invariant joining μ of the invariant measures on $\Gamma\backslash G$ and $\Lambda\backslash H$; it's not hard to see that μ is $(u, \tilde{u})$-invariant, i.e., μ is a joining of the translation by u on $\Gamma\backslash G$ and the translation by $\tilde{u}$ on $\Lambda\backslash H$. Let (Ω, ω) be (almost) any ergodic component of the translation by $(u, \tilde{u})$ on (GRAPH, μ); Lemma 2.2 asserts ω is a joining of the invariant measures on $\Gamma\backslash G$ and $\Lambda\backslash H$.

Where convenient (and relatively harmless) we ignore null sets. For example, there is a conull u-invariant subset $\mathcal{S}$ of $\Gamma\backslash G$ such that, for all $s \in \mathcal{S}$ and all $t \in \Lambda\backslash H$ with $s\psi = tM$, one has $su\psi = tM\tilde{u}$; we pretend $\mathcal{S} = \Gamma\backslash G$.

Step 1. *$\tilde{u}$ centralizes M.* This is a direct consequence of Lemma 2.10.

Step 2. *We may assume there is no proper subgroup M' of M such that each fiber of ω over $\Gamma\backslash G$ is supported on a single M'-orbit (a.e.).* If there is such a subgroup, then by the descending chain condition on compact subgroups of H, there is a minimal such subgroup, say M_0; under the natural quotient map $\Gamma\backslash G \times \Lambda\backslash H \to \Gamma\backslash G \times \Lambda\backslash H/M_0$, the measure ω pushes to a joining ω', and (almost) every fiber of ω' over $\Gamma\backslash G$ is supported on a single point: this means ω' is the joining associated to a map $\psi': \Gamma\backslash G \to \Lambda\backslash H/M_0$; this map conjugates the translation by u on $\Gamma\backslash G$ to the translation by $\tilde{u}$ on $\Lambda\backslash H/M_0$. We can replace ψ with ψ', and M with M_0; the minimality of M_0 implies it has no proper subgroup M' of the specified type.

Step 3 (cf. [**13**, Lemma 3.1]). *ψ is affine for $C_G(u)^0$ via $C_H(\tilde{u})^0$.* Let c be any small element of $C_G(u)$. The polynomial divergence of $\tilde{u}$-orbits on H/M can be used, in the style of M. Ratner [7, Lemma 3.2] (see also [**13**, Lemma 3.1] and [**2**]), to show, for a.e. $s \in \Gamma\backslash G$, that for any $t \in \Lambda\backslash H$ with $(s, t) \in$ GRAPH, there is some small $\tilde{c}_{s,t} \in C_H(\tilde{u})$ with $(sc, s\tilde{c}_{s,t}) \in$ GRAPH. (Note that $\tilde{c}_{s,t}$ is unique (mod M) because $C_H(\tilde{u})$ is essentially free on $\Lambda\backslash H$ [**13**, Lemma 2.8].)

We wish to find some $\tilde{c} \in N_H(M)$ such that $\tilde{c}_{s,t} \in \tilde{c}M$ for (almost) all $(s, t) \in$ GRAPH. For $(s, t) \in$ GRAPH, it follows from the fact that GRAPH is invariant under translation by $(u, \tilde{u})$, that both $(suc, t\tilde{u}\tilde{c}_{su,t\tilde{u}})$ and $(scu, t\tilde{c}_{s,t}\tilde{u})$ are in GRAPH. Since $scu = suc$ and $t\tilde{c}_{s,t}\tilde{u} = t\tilde{u}\tilde{c}_{s,t}$, then the uniqueness of $\tilde{c}_{su,t\tilde{u}}$ implies $\tilde{c}_{su,t\tilde{u}} \in \tilde{c}_{s,t}M$; it immediately follows that $\tilde{c} = \tilde{c}_{s,t}$ is (essentially) constant (mod M) on almost any ergodic component, such as (Ω, ω), of μ. A simple calculation shows, for $k \in M$, that $\tilde{c}_{s,tk} \in k^{-1}\tilde{c}_{s,t}M$; if both (s, t) and (s, tk) are in the support of the fiber of ω over s, this implies $\tilde{c}k\tilde{c}^{-1} \in M$; and hence $k \in M \cap (\tilde{c}M\tilde{c}^{-1})$. Then Step 2 implies $M \cap \tilde{c}^{-1}M\tilde{c}$ is not a proper subgroup of M; hence $\tilde{c} \in N_H(M)$.

For (almost) any $(s, t) \in$ GRAPH, there is some $k \in M$ such that (s, tk) is in the support of the fiber of ω over s; hence $\tilde{c}_{s,tk} \in \tilde{c}M$. Therefore $\tilde{c}_{s,t} \in k^{-1}\tilde{c}_{s,tk}M = k^{-1}\tilde{c}M = \tilde{c}M$, as desired.

Definition. Let $\mathrm{Stab}_G(\psi) = \{g \in G \mid sg\psi = s\psi$ for a.e. $s \in \Gamma\backslash G\}$ and $\mathrm{Aff}_G(\psi) = \{g \in G \mid \psi$ is affine for $g\}$. Note that $\mathrm{Stab}_G(\psi) \lhd \mathrm{Aff}_G(\psi)$, because the kernel of a homomorphism is always a normal subgroup.

Step 4. *We may assume* $\mathrm{Stab}_G(\psi)$ *is compact, and that the fibers of* $\bar{\psi}: \Gamma\backslash G/\mathrm{Stab}_G(\psi) \to \Lambda\backslash H/M$ *are finite (a.e.).* The Mautner phenomenon implies there is a normal subgroup N of G, contained in $\mathrm{Stab}_G(\psi)$, such that $\mathrm{Stab}_G(\psi)/N$ is compact (see Lemma 2.5); replacing G with G/N, we may assume $N = e$, so $\mathrm{Stab}_G(\psi)$ is compact. Now, following an idea of M. Ratner [**9**, Theorem 3, and **8**, Lemma 3.1], much as in the proofs of Lemmas 7.4 and 7.5 of [**13**], we can use the shearing nature of unipotent flows (Corollary 2.13) to show that the fibers of $\bar{\psi}$ are finite (a.e.), i.e., that there is a conull subset $\mathscr{S}$ of $\Gamma\backslash G/\mathrm{Stab}_G(\psi)$ such that, for all $t \in \Lambda\backslash H/M$, the inverse image $(t\bar{\psi}^{-1}) \cap \mathscr{S}$ is finite.

Remark. If each fiber of $\bar{\psi}$ is a single point, then $\bar{\psi}$ is one-to-one, i.e., invertible; its inverse induces a map $\hat{\psi}^{-1}: \Lambda\backslash H \to \Gamma\backslash G/\mathrm{Stab}_G(\psi)$ that conjugates the translation by $\tilde{u}$ on $\Lambda\backslash H$ to the translation by u on $\Gamma\backslash G/\mathrm{Stab}_G(\psi)$; anything we've proved about ψ must also be true for $\hat{\psi}^{-1}$ (after allowing for the interchange of G and H, and so forth). This observation will be crucial in Steps 5, 6, and 7 of the proof; to salvage this observation for the case where the fibers of $\bar{\psi}$ are not assumed to be single points, we need to allow $\hat{\psi}^{-1}$ to be a joining with finite fibers over $\Lambda\backslash H$; it is for this reason that we need the more general hypotheses of Theorem 4.1′. But, for simplicity, I will pretend $\bar{\psi}$ is invertible and ignore the need for finite-fiber joinings.

Step 5. *For any unipotent element* v *of* $C_G(u)^0$, *there is a unipotent element* $\tilde{v}$ *of* $N_H(M)$, *with* $sv\psi = s\psi \cdot \tilde{v}$ *for a.e.* $s \in \Gamma\backslash G$. For convenience, let $M_0 = M \cap C_H(\tilde{u})^0$. Step 3 provides us with a map $\sim: C_G(u)^0 \to (C_H(\tilde{u}) \cap N_H(M))/M$. Step 4 (or the subsequent remark) asserts that $\bar{\psi}$ is invertible, and there is a corresponding map $\hat{\psi}^{-1}: \Lambda\backslash H \to \Gamma\backslash G/\mathrm{Stab}_G(\psi)$; applying Step 3 to $\hat{\psi}^{-1}$ provides us with a map $C_H(\tilde{u})^0/M_0 \to C_G(u)/\mathrm{Stab}_G(\psi)$; this map is an inverse to $\sim$; we conclude that $M_0 \lhd C_H(\tilde{u})^0$ and that

$$\sim: \frac{C_G(u)^0}{\mathrm{Stab}_G(\psi) \cap C_G(u)^0} \to \frac{C_H(\tilde{u})^0}{M_0}$$

is an isomorphism. Now Lemma 2.9 asserts that, for any unipotent element v of G belonging to $C_G(u)^0$, we can choose $\tilde{v}$ to be unipotent, as desired.

Step 6 [**13**, Lemma 7.3]. $\mathrm{Stab}_G(\psi) \lhd G$, *so we may assume* $\mathrm{Stab}_G(\psi) = e$. Let U be the identity component of any maximal unipotent subgroup of G that contains u. Using the Moore Ergodicity Theorem (cf. [**13**, Theorem 2.14]), it is easy to find an element v in the center of U such that v is ergodic on $\Gamma\backslash G$. Step 3 implies ψ is affine for v, and Step 5 asserts that we can choose $\tilde{v}$ to be unipotent; then the hypotheses of Theorem 4.1 are satisfied if we put v in the place of u and $\tilde{v}$ in the place of $\tilde{u}$; in this situation, Step 3 implies ψ is affine for $C_G(Z(U))^0$; in particular, ψ is affine for U and for the maximal compact factor K of G. Since $\mathrm{Stab}_G(\psi) \lhd \mathrm{Aff}_G(\psi)$, this implies $\mathrm{Stab}_G(\psi)$ is normalized by U and K. Because $\mathrm{Stab}_G(\psi)$ is compact (Step 4), it follows that $\mathrm{Stab}_G(\psi)$ is a normal subgroup of G (see Lemma 2.6), and as there is no loss in modding it out, we may assume $\mathrm{Stab}_G(\psi) = e$.

Step 7. *$M = e$, so Theorem* 1.4 *asserts ψ is affine* (*a.e.*), *as desired.* Step 4 (or the subsequent remark), together with Step 6, implies that ψ is invertible. Then $\psi^{-1}: \Lambda\backslash H/M \to \Gamma\backslash G$ induces a map $\hat{\psi}^{-1}: \Lambda\backslash H \to \Gamma\backslash G$ that is affine for $\tilde{u}$ via u; now $\mathrm{Stab}_H(\hat{\psi}^{-1}) = M$, so, with $\hat{\psi}^{-1}$ in the role of ψ, Step 6 asserts $M \lhd H$ - but M contains no nontrivial normal subgroup of H, so this implies $M = e$. Hence $\psi: \Gamma\backslash G \to \Lambda\backslash H$ satisfies the hypotheses of Theorem 1.4, which implies that ψ is affine. □

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