

Real Representations of Semisimple Lie Algebras Have \mathbb{Q} -forms

Dave Morris*

To Professor M. S. Raghunathan on his sixtieth birthday

Abstract

We prove that each real semisimple Lie algebra \mathfrak{g} has a \mathbb{Q} -form $\mathfrak{g}_{\mathbb{Q}}$, such that every real representation of $\mathfrak{g}_{\mathbb{Q}}$ can be realized over \mathbb{Q} . This was previously proved by M. S. Raghunathan (and rediscovered by P. Eberlein) in the special case where \mathfrak{g} is compact.

1 Introduction

All Lie algebras and all representations are assumed to be finite-dimensional. It is easy to see, from the theory of highest weights, that if \mathfrak{g} is an \mathbb{R} -split, semisimple Lie algebra over \mathbb{R} , then every \mathbb{C} -representation of \mathfrak{g} has an \mathbb{R} -form (see 3.1). (That is, if $V_{\mathbb{C}}$ is a representation of \mathfrak{g} over \mathbb{C} , then there is a real representation V of \mathfrak{g} , such that $V_{\mathbb{C}} \cong V \otimes_{\mathbb{R}} \mathbb{C}$.) Because every semisimple Lie algebra over \mathbb{C} has an \mathbb{R} -split real form, this leads to the following immediate conclusion:

Remark 1.1 Any complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ has a real form \mathfrak{g} , such that every \mathbb{C} -representation of \mathfrak{g} has a real form.

In this paper, we prove the analogous statement with the field extension \mathbb{C}/\mathbb{R} replaced with \mathbb{R}/\mathbb{Q} .

Theorem 1.2 (see 2.6(2)) *Any real semisimple Lie algebra \mathfrak{g} has a \mathbb{Q} -form $\mathfrak{g}_{\mathbb{Q}}$, such that every real representation of $\mathfrak{g}_{\mathbb{Q}}$ has a \mathbb{Q} -form.*

*Former name: Dave Witte.

In the special case where \mathfrak{g} is compact, the theorem was proved by M. S. Raghunathan [R2, §3]. This special case was independently rediscovered by P. Eberlein [E], and a very nice proof was found by R. Pink and G. Prasad (personal communication, see §4). When \mathfrak{g} is compact, these authors showed that the “obvious” \mathbb{Q} -form of \mathfrak{g} has the desired property.

At the other extreme, where \mathfrak{g} is \mathbb{R} -split, we may take $\mathfrak{g}_{\mathbb{Q}}$ to be any \mathbb{Q} -split \mathbb{Q} -form of \mathfrak{g} (see 3.1).

The general case is a combination of the two extremes, and the desired \mathbb{Q} -form can be obtained from a Chevalley basis of $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ by slightly modifying a construction of A. Borel [B] (see §6). We give two different proofs that this \mathbb{Q} -form has the desired property: one proof is by the method of Pink and Prasad, using a little bit of number theory (see §4), and the other proof is by reducing to the compact case, so Raghunathan’s theorem applies (see §5).

It would be interesting to characterize the semisimple Lie algebras $\mathfrak{g}_{\mathbb{Q}}$ over \mathbb{Q} , such that every real representation has a \mathbb{Q} -form. For example, work of J. Tits [T] implies that every \mathbb{Q} -form of $\mathfrak{sp}(n)$ has this property (see 7.2). On the other hand, it is important to note that there exist examples of $\mathbb{Q}(i)$ -split Lie algebras that do not have this property (see 7.4). (Real representations of such a Lie algebra can be realized over both $\mathbb{Q}(i)$ and \mathbb{R} , but not over $\mathbb{Q}(i) \cap \mathbb{R} = \mathbb{Q}$.)

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2 More Precise Statement of the Main Result

Proposition 2.1 (Pink, Prasad, see §4) *Suppose G is a connected, reductive algebraic \mathbb{Q} -group. If*

- a. G is split over some imaginary quadratic extension F of \mathbb{Q} , and
 - b. G is quasi-split over the p -adic field \mathbb{Q}_p , for every odd prime p ,
- then each irreducible \mathbb{Q} -representation of G remains irreducible over \mathbb{R} .

This can be restated in the following equivalent form.

Definition 2.2 Suppose (π, V) is a real representation of an algebraic \mathbb{Q} -group G . A \mathbb{Q} -subspace $V_{\mathbb{Q}}$ of $V_{\mathbb{R}}$ is a \mathbb{Q} -form of (π, V) if

- $V_{\mathbb{Q}}$ is $G_{\mathbb{Q}}$ -invariant, and
- $V_{\mathbb{Q}}$ is the \mathbb{Q} -span of an \mathbb{R} -basis of $V_{\mathbb{R}}$ (so $V_{\mathbb{R}} \cong V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$).

Corollary 2.3 (see 3.3) *If G is as in Proposition 2.1, then every real representation of G has a \mathbb{Q} -form.*

Proposition 2.4 (see §6) *Every connected, simply connected, semisimple real algebraic group has a \mathbb{Q} -form satisfying the hypotheses of Proposition 2.1.*

Combining Proposition 2.4 with Proposition 2.1 and Corollary 2.3 immediately yields the following conclusion.

Definition 2.5 Suppose

- $\mathfrak{g}_{\mathbb{Q}}$ is a Lie algebra over \mathbb{Q} , and
- (π, V) is a real representation of $\mathfrak{g}_{\mathbb{Q}}$.

A \mathbb{Q} -subspace $V_{\mathbb{Q}}$ of V is a \mathbb{Q} -form of (π, V) if

- $V_{\mathbb{Q}}$ is $\mathfrak{g}_{\mathbb{Q}}$ -invariant, and
- $V_{\mathbb{Q}}$ is the \mathbb{Q} -span of an \mathbb{R} -basis of V (so $V \cong V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$).

Corollary 2.6 *Any real semisimple Lie algebra \mathfrak{g} has a \mathbb{Q} -form $\mathfrak{g}_{\mathbb{Q}}$, such that*

1. *if $V_{\mathbb{Q}}$ is any irreducible \mathbb{Q} -representation of $\mathfrak{g}_{\mathbb{Q}}$, then the \mathbb{R} -representation $V_{\mathbb{R}} = V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$ is irreducible; and*
2. *every real representation of $\mathfrak{g}_{\mathbb{Q}}$ has a \mathbb{Q} -form.*

See Theorem 5.2 for a version of Proposition 2.1 that replaces 2.1(b) with a quite different hypothesis, due to M. S. Raghunathan. The \mathbb{Q} -form constructed in §6 also satisfies this alternate hypothesis, so this yields a different proof of Corollary 2.6.

3 Preliminaries

The following is well known (see, for example, [T, Theorem 2.5]).

Lemma 3.1 *Let F be a subfield of \mathbb{C} , \mathfrak{g} be a semisimple Lie algebra over F , and $V_{\mathbb{C}}$ be a \mathbb{C} -representation of \mathfrak{g} . If \mathfrak{g} is F -split, then V has an F -form.*

Proof Let \mathfrak{t} be a maximal F -split torus of \mathfrak{g} . Because every representation of \mathfrak{g} is a direct sum of irreducibles, we may assume $V_{\mathbb{C}}$ is irreducible; let λ be the highest weight of $V_{\mathbb{C}}$ (with respect to some ordering of the roots of \mathfrak{t}). Since λ is a character of the F -split torus \mathfrak{t} , we know that $\lambda(\mathfrak{t}_F) \subset F$. So there is an F -representation V_F of \mathfrak{g} with highest weight λ . Hence, $V_F \otimes_F \mathbb{C} \cong V_{\mathbb{C}}$, so V_F is (isomorphic to) an F -form of $V_{\mathbb{C}}$. □

For future reference, let us record the following consequence of this fact.

Corollary 3.2 *Suppose, for some quadratic extension F of \mathbb{Q} , that \mathfrak{g} is an F -split, semisimple Lie algebra over \mathbb{Q} .*

1. *If V_F is any irreducible F -representation of \mathfrak{g}_F , then $V_{\mathbb{C}}$ is irreducible.*
2. *If $V_{\mathbb{Q}}$ is any irreducible \mathbb{Q} -representation of \mathfrak{g} , then $V_{\mathbb{C}}$ is either irreducible or the direct sum of two irreducibles.*

Proof (1) The proof of Lemma 3.3(\Leftarrow) below shows that this is a consequence of Lemma 3.1.

(2) Write $F = \mathbb{Q}[\sqrt{r}]$, for some $r \in \mathbb{Q}$. Because $V_F = V_{\mathbb{Q}} + \sqrt{r}V_{\mathbb{Q}}$ (and $V_{\mathbb{Q}}$ is an irreducible $\mathfrak{g}_{\mathbb{Q}}$ -module), we know that the \mathfrak{g}_F -module V_F is either irreducible or the direct sum of two irreducibles. Then the desired conclusion follows from (1). □

The following observation must be well known. The direction (\Rightarrow) can be found in [R2, §3].

Lemma 3.3 *Suppose G is a connected, semisimple algebraic group over \mathbb{Q} . Every irreducible \mathbb{Q} -representation of G remains irreducible over \mathbb{R} if and only if every real representation of G has a \mathbb{Q} -form.*

Proof (\Leftarrow) Let V be a \mathbb{Q} -representation of G , such that $V_{\mathbb{R}}$ is reducible. (We wish to show that V is reducible.) We may write $V_{\mathbb{R}} = U_1 \oplus U_2$, for

some nontrivial \mathbb{R} -representations U_1 and U_2 . By assumption, there exist \mathbb{Q} -representations V_1 and V_2 , such that $(V_j)_{\mathbb{R}} \cong U_j$. Then

$$V_{\mathbb{R}} = U_1 \oplus U_2 \cong (V_1)_{\mathbb{R}} \oplus (V_2)_{\mathbb{R}} \cong (V_1 \oplus V_2)_{\mathbb{R}}.$$

Thus, V is isomorphic to $V_1 \oplus V_2$ over \mathbb{R} . Since both V and $V_1 \oplus V_2$ are defined over \mathbb{Q} , this implies that V is isomorphic to $V_1 \oplus V_2$ over \mathbb{Q} . (The \mathfrak{g} -equivariant maps from $V_{\mathbb{R}}$ to $(V_1 \oplus V_2)_{\mathbb{R}}$ form a real vector space that is defined over \mathbb{Q} , so the \mathbb{Q} -points span.) Thus, V is reducible.

(\Rightarrow) Let V be a real representation of G . Because representations of G are completely reducible, we may assume that V is irreducible. To simplify notation (and because this is the only case we need), let us also assume that G is split over some imaginary quadratic extension F of \mathbb{Q} . Then $V_{\mathbb{C}}$ has an F -form U . Let $U|_{\mathbb{Q}}$ be the \mathbb{Q} -representation obtained by viewing U as a vector space over \mathbb{Q} .

Now write $U|_{\mathbb{Q}} = U_1 \oplus \cdots \oplus U_r$ as a direct sum of irreducible \mathbb{Q} -modules. Then

$$V_{\mathbb{C}}|_{\mathbb{R}} = U_{\mathbb{C}}|_{\mathbb{R}} \cong (U|_{\mathbb{Q}})_{\mathbb{R}} \cong (U_1)_{\mathbb{R}} \oplus \cdots \oplus (U_r)_{\mathbb{R}}.$$

Since V is a submodule of $V_{\mathbb{C}}|_{\mathbb{R}}$ (indeed, $V_{\mathbb{C}}|_{\mathbb{R}}$ is the direct sum $V \oplus iV$ of two copies of V), and, by assumption, each $(U_j)_{\mathbb{R}}$ is irreducible, we conclude that V is isomorphic to $(U_j)_{\mathbb{R}}$, for some j . So (up to isomorphism) U_j is a \mathbb{Q} -form of V . □

We also use the following (special case of a) result of J. Tits [T, Theorems 7.2(i) and 3.3] that applies to the quasi-split case. In our applications, F will be a p -adic field \mathbb{Q}_p (see 2.1(b)).

Proposition 3.4 (Tits) *Let F be a field of characteristic zero, G be a connected, reductive algebraic F -group, and (π, V) be an irreducible F -representation of G . If G is quasi-split over F , then $\text{End}_G(V)$ is commutative.*

Proof By assumption, G has a Borel subgroup B that is defined over F . Let $v \in V$ be a nonzero vector that is fixed by every element of the unipotent radical of B , let \bar{F} be the algebraic closure of F , and let $V_{\bar{F}} = V \otimes_F \bar{F}$.

Schur's Lemma asserts that $\text{End}_G(V)$ is a division algebra. By enlarging F , we may assume that F is the center of $\text{End}_G(V)$, so $\text{End}_G(V_{\bar{F}}) = \text{End}_G(V) \otimes_F \bar{F}$ is a simple algebra, which implies that $V_{\bar{F}}$ is isotypic: we have $V_{\bar{F}} = W \oplus \cdots \oplus W$, for some irreducible $\mathfrak{g}_{\bar{F}}$ -module W . This implies that v is a weight vector (that is, an eigenvector for $\pi(B)$), so V is a highest-weight module. Therefore $\text{End}_G(V) = F$, which is abelian, as desired. □

The method of R. Pink and G. Prasad utilizes the following classical result of number theory. It is obtained by combining the Hasse Principal (or local-to-global principle) with the fact that the sum of the local invariants of a quaternion algebra (or, what is the same thing, of a quadratic form) is 0 (so, if all but one of them vanish, then they all must vanish).

Lemma 3.5 (cf. [Se, Corollary 3.2.3, p. 43]) *Let \mathcal{C} be a quaternion division algebra over \mathbb{Q} . If \mathcal{C} splits over \mathbb{Q}_p , for every odd prime p , then \mathcal{C} does **not** split over \mathbb{R} .*

Remark 3.6 The exceptional prime 2 can be replaced with any other prime in Lemma 3.5: if there is a prime p_0 , such that G is quasi-split over the p -adic field \mathbb{Q}_p , for every prime $p \neq p_0$, then \mathcal{C} does **not** split over \mathbb{R} . But we have no need for this more general (and somewhat less concise) version.

4 The Method of Pink and Prasad

Proof of Proposition 2.1 Suppose $(\pi, V_{\mathbb{Q}})$ is an irreducible representation of G over \mathbb{Q} . Let

$$\mathcal{C} = \text{End}_G(V_{\mathbb{Q}})$$

be the centralizer of $\pi(G_{\mathbb{Q}})$ in $\text{End}_{\mathbb{Q}}(V_{\mathbb{Q}})$. Then Schur's Lemma tells us that \mathcal{C} is a division algebra over \mathbb{Q} .

Because G splits over the quadratic extension F , we know that $V_{\mathbb{C}}$ is either irreducible or the sum of two irreducibles (see 3.2(2)).

Case 1. Assume $V_{\mathbb{C}}$ is irreducible. In this case $V_{\mathbb{R}}$ is obviously irreducible.

Case 2. Assume $V_{\mathbb{C}}$ is the direct sum of two irreducibles that are not isomorphic. From the assumption of this case, and the fact that G splits over F , we know that V_F is the direct sum of two irreducibles that are not isomorphic. Therefore, $\text{End}_G(V_F) \cong F \oplus F$, so $\mathcal{C} \otimes_{\mathbb{Q}} F \cong F \oplus F$.

Write $F = \mathbb{Q}[\sqrt{-r}]$, for some $r \in \mathbb{Q}^+$.

- Because $F \oplus F$ is commutative, we know that \mathcal{C} is commutative, so \mathcal{C} is a field.
- Because $\dim_F(F \oplus F) = 2$, we know that $\dim_{\mathbb{Q}} \mathcal{C} = 2$.
- Because $\mathcal{C} \otimes_{\mathbb{Q}} F = \mathcal{C}[\sqrt{-r}]$ is not a field, we know that \mathcal{C} contains a root of $x^2 + r$.

We conclude that $\mathcal{C} \cong F$. Therefore

$$\text{End}_G(V_{\mathbb{R}}) \cong \mathcal{C} \otimes_{\mathbb{Q}} \mathbb{R} \cong F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{C}$$

is a field. So $V_{\mathbb{R}}$ is irreducible.

Case 3. Assume $V_{\mathbb{C}}$ is the direct sum of two irreducibles that are isomorphic. In this case, we know that $\text{End}_G(V_{\mathbb{C}}) \cong \text{Mat}_{2 \times 2}(\mathbb{C})$ is 4-dimensional over \mathbb{C} . Since $\text{End}_G(V_{\mathbb{C}}) \cong \mathcal{C} \otimes_{\mathbb{Q}} \mathbb{C}$, we conclude that \mathcal{C} is 4-dimensional over \mathbb{Q} . Thus, \mathcal{C} is a quaternion algebra over \mathbb{Q} .

For every odd prime p , Proposition 3.4 (and the fact that $\mathcal{C} \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is not commutative) implies that $V_{\mathbb{Q}_p}$ is reducible, so $\mathcal{C} \otimes_{\mathbb{Q}} \mathbb{Q}_p = \text{End}_G(V_{\mathbb{Q}_p})$ is not a division algebra. In other words, \mathcal{C} splits over \mathbb{Q}_p .

Now, Lemma 3.5 asserts that \mathcal{C} does not split over \mathbb{R} . This means that $\text{End}_G(V_{\mathbb{R}}) \cong \mathcal{C} \otimes_{\mathbb{Q}} \mathbb{R}$ is a division algebra. We conclude that $V_{\mathbb{R}}$ is irreducible, as desired. □

Remark 4.1 From the proof (and Rem. 3.6), it is clear that the exceptional prime 2 can be replaced with any other prime in Condition 2.1(b): it suffices to assume that there is a prime p_0 , such that G is quasi-split over the p -adic field \mathbb{Q}_p , for every prime $p \neq p_0$. The case $p_0 = 2$ is all we need for our proof of Corollary 2.6.

5 Reducing to the Compact Case

Definition 5.1 Suppose G is a connected, reductive algebraic \mathbb{Q} -group. Let

- S be a maximal \mathbb{Q} -split torus of G ;
- $C = C_G(S)$ be the centralizer of S ;
- M' be the (unique) maximal connected, semisimple subgroup of the reductive group C ;
- T be a maximal \mathbb{Q} -torus of M' ;
- Φ^+ be the positive roots of $(\mathfrak{m}'_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ (with respect to some ordering); and
- Φ^- be the set of negative roots.

We call M' the *semisimple anisotropic kernel* of G .

We say that *the longest element of the Weyl group of the anisotropic kernel of G is realized over \mathbb{Q}* if there is some $w \in N_{M'}(T)_{\mathbb{Q}}$, such that

$$w(\Phi^+) = \Phi^-.$$

Here, as usual, the normalizer $N_{M'}(T)$ acts on \mathfrak{t}^* by

$$w(\lambda)(t) = \lambda(w^{-1}tw).$$

It is important to notice (from the subscripts in $N_{M'}(T)_{\mathbb{Q}}$) that w is required to be in the semisimple group M' , and that w is required to be a \mathbb{Q} -element.

Theorem 5.2 *Suppose G is a connected, reductive algebraic \mathbb{Q} -group. If*

- a. G is split over some imaginary quadratic extension F of \mathbb{Q} ,
- b. \mathbb{Q} -rank $G = \mathbb{R}$ -rank G , and
- c. the longest element of the Weyl group of the anisotropic kernel of G is realized over \mathbb{Q} ,

then each irreducible \mathbb{Q} -representation of G remains irreducible over \mathbb{R} .

M.S. Raghunathan [R2, §3] proved Theorem 5.2 in the special case where G is semisimple, and \mathbb{R} -rank $G = 0$ (in other words, G is compact). The following result shows that the general case follows from this.

Proposition 5.3 *Suppose G is a connected, reductive algebraic group over \mathbb{Q} , and let M' be the semisimple anisotropic kernel of G . Suppose that G is split over some imaginary quadratic extension F of \mathbb{Q} , \mathbb{Q} -rank $G = \mathbb{R}$ -rank G , and every irreducible \mathbb{Q} -representation of M' remains irreducible over \mathbb{R} . Then every irreducible \mathbb{Q} -representation of G remains irreducible over \mathbb{R} .*

Remark 5.4 There is no need to assume that the quadratic extension F is imaginary in (5.2) or (5.3): if F is real, then the hypotheses imply that G is \mathbb{Q} -split, so Lemma 3.1 applies.

Before proving the proposition, let us state a simple lemma, which reduces the construction of \mathbb{Q} -forms of representations of G to the same problem for certain representations of a minimal parabolic P . It is similar to the usual construction of highest weight modules.

Lemma 5.5 *Let \mathfrak{g} be a semisimple Lie algebra over \mathbb{Q} , \mathfrak{t} be a maximal \mathbb{Q} -split torus of \mathfrak{g} , $\Phi_{\mathbb{Q}}$ be the system of \mathbb{Q} -roots of $(\mathfrak{g}, \mathfrak{t})$, \mathfrak{p} be a minimal parabolic \mathbb{Q} -subalgebra of \mathfrak{g} that contains \mathfrak{t} , V be an irreducible, real \mathfrak{g} -module, and $\lambda : \mathfrak{t} \rightarrow \mathbb{Q}$ be the highest weight of V , with respect to the ordering of $\Phi_{\mathbb{Q}}$ determined by the parabolic \mathfrak{p} . If there exists a \mathfrak{p} -invariant \mathbb{Q} -form of the weight space V^{λ} , then the representation V has a \mathbb{Q} -form.*

Proof Let W be a \mathfrak{p} -invariant \mathbb{Q} -form of V^{λ} , let $\Delta_{\mathbb{Q}}$ be the base of $\Phi_{\mathbb{Q}}$ determined by \mathfrak{p} , and let \widetilde{W} be the \mathbb{Q} -span of

$$\{y_1 y_2 \cdots y_k w \mid w \in W, \alpha_j \in \Delta_{\mathbb{Q}}, y_j \in \mathfrak{g}_{-\alpha_j}, k \in \mathbb{N}\}. \quad (5.1)$$

Step 1. \widetilde{W} is \mathfrak{g} -invariant. From the definition of \widetilde{W} , it is obvious that $[\mathfrak{g}_{-\alpha}, \widetilde{W}] \subset \widetilde{W}$ for all $\alpha \in \Delta_{\mathbb{Q}}$. Also, because both the centralizer $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{t})$ and

the root space \mathfrak{g}_α are contained in \mathfrak{p} , it is not difficult to see (by induction on k) that $[\mathfrak{c}_\mathfrak{p}(t), \widetilde{W}] \subset \widetilde{W}$ and $[\mathfrak{g}_\alpha, \widetilde{W}] \subset \widetilde{W}$, for all $\alpha \in \Delta_{\mathbb{Q}}$. Since

$$\mathfrak{c}_\mathfrak{p}(t) \cup \bigcup_{\alpha \in \Delta_{\mathbb{Q}}} (\mathfrak{g}_\alpha \cup \mathfrak{g}_{-\alpha}) \text{ generates } \mathfrak{g},$$

we conclude that \widetilde{W} is \mathfrak{g} -invariant.

Step 2. \widetilde{W} spans V over \mathbb{R} . The \mathbb{R} -span of \widetilde{W} is a submodule of V , so the desired conclusion follows from the fact that V is irreducible.

Step 3. If a_1, \dots, a_r are real numbers that are linearly independent over \mathbb{Q} , and w_1, \dots, w_r are nonzero elements of \widetilde{W} , then $\sum_{j=1}^r a_j w_j \neq 0$. Suppose $\sum_{j=1}^r a_j w_j = 0$. (This will lead to a contradiction.)

Since $y_1 y_2 \cdots y_k w \in V^{\lambda - \alpha_1 - \cdots - \alpha_k}$ for w, k and y_j as in (5.1), we see that

$$\widetilde{W} \cap V^\lambda = W \tag{5.2}$$

and

$$\widetilde{W} = \bigoplus_{\mu \in \mathfrak{t}^*} (\widetilde{W} \cap V^\mu). \tag{5.3}$$

Because of (5.3), we may assume there is some weight μ , such that $w_j \in V^\mu$ for all j . (Project to some V^μ , and delete the w_j 's whose projection is 0.)

Because V is irreducible, and λ is the highest weight, there exist $m \in \mathbb{N}$ and $x_1, \dots, x_m \in \bigcup_{\alpha \in \Delta_{\mathbb{Q}}} \mathfrak{g}_\alpha$, such that $x_1 \cdots x_m w_1$ is a nonzero element of V^λ . From (5.3), we see that $x_1 \cdots x_m w_j \in V^\lambda$ for every j . Hence, (5.2) implies that $x_1 \cdots x_m w_j \in W$ for all j . Since

$$\sum_{j=1}^r a_j (x_1 \cdots x_m w_j) = x_1 \cdots x_m \sum_{j=1}^r a_j w_j = x_1 \cdots x_m \cdot 0 = 0,$$

and W is a \mathbb{Q} -form of V^λ , this implies that $x_1 \cdots x_m w_j = 0$ for every j . This contradicts the choice of x_1, \dots, x_m .

Step 4. Completion of proof. From Steps 2 and 3, we see that the natural scalar-multiplication map $\mathbb{R} \otimes_{\mathbb{Q}} \widetilde{W} \rightarrow V$ is a bijection. So \widetilde{W} is a \mathbb{Q} -form of the vector space V . By combining this with Step 1, we conclude that \widetilde{W} is a \mathbb{Q} -form of the representation. □

Proof of Proposition 5.3 We consider two cases.

Case 1. Assume that the semisimple part of $G_{\mathbb{R}}$ is compact. We may write $G = M'AT$, where M' is connected and semisimple (so, by assumption, $M'_{\mathbb{R}}$ is compact), A is a \mathbb{Q} -split torus and T is a torus that is anisotropic over \mathbb{Q} . Let V be an irreducible \mathbb{Q} -representation of G . Since A , being a \mathbb{Q} -split, central torus, acts by scalars, we know that V is an irreducible \mathbb{Q} -representation of $M'T$.

Subcase 1.1. Assume T acts trivially on V . Then V is an irreducible \mathbb{Q} -representation of M' , so, by assumption, it remains irreducible over \mathbb{R} .

Subcase 1.2. Assume T acts nontrivially on V . Because T_F is an F -split, central torus, its Lie algebra defines an action of F on V that centralizes M' . Thus, we may think of V as an irreducible F -representation of M' . Let us say $V_{\mathbb{Q}} = W_F$; then $V_{\mathbb{R}} = W_{\mathbb{C}}$. Because M' is F -split, we know that $W_{\mathbb{C}}$ is irreducible (see 3.2(1)).

Because \mathbb{Q} -rank $T = 0$ and \mathbb{Q} -rank $G = \mathbb{R}$ -rank G , we see that \mathbb{R} -rank $T = 0$, which means $T_{\mathbb{R}}$ is compact, so the Lie algebra of T acts by purely imaginary scalars on $W_{\mathbb{C}}$. Thus, any $M'T$ -invariant \mathbb{R} -submodule of $V_{\mathbb{R}}$ is an M' -invariant \mathbb{C} -submodule of $W_{\mathbb{C}}$. Hence, the conclusion of the preceding paragraph implies that $V_{\mathbb{R}}$ is irreducible.

Case 2. The general case. Given a representation V of G over \mathbb{R} , we wish to show that V has a \mathbb{Q} -form (see Lemma 3.3). Because representations of G are completely reducible, we may assume that V is irreducible.

Let P be a minimal parabolic \mathbb{Q} -subgroup of G , let T be a maximal \mathbb{Q} -split torus of P , and λ be the highest weight of V (with respect to T and P).

Now V^{λ} is $C_G(T)$ -invariant, so, from Case 1, we know that the vector space V^{λ} has a \mathbb{Q} -form U that is $C_G(T)_{\mathbb{Q}}$ -invariant. Then, since the unipotent radical of P acts trivially on V^{λ} , we know that U is $P_{\mathbb{Q}}$ -invariant. So Lemma 5.5 implies that V has a \mathbb{Q} -form. □

6 Construction of a Good \mathbb{Q} -form

In this section, we provide an explicit construction of a \mathbb{Q} -form of G that satisfies the hypotheses of both Proposition 2.1 and Theorem 5.2. Our main theorem (1.2) could be obtained from a \mathbb{Q} -form that satisfies the hypotheses of either one of these results, but it is interesting to note that we are able to satisfy both of them simultaneously.

Proposition 6.1 *If G is a connected, simply connected, semisimple algebraic \mathbb{R} -group, then G has a \mathbb{Q} -form, such that*

1. G is split over $\mathbb{Q}(i)$,
2. G is quasi-split over the p -adic field \mathbb{Q}_p , for every odd prime p ,
3. \mathbb{Q} -rank $G = \mathbb{R}$ -rank G , and
4. every element of the Weyl group of the anisotropic kernel of G is realized over \mathbb{Q} .

The argument is a straightforward adaptation of A. Borel’s [B] classical proof of the existence of an anisotropic \mathbb{Q} -form.

Actually, like Borel, we do not directly construct $G_{\mathbb{Q}}$ itself, but only the Lie algebra $\mathfrak{g}_{\mathbb{Q}}$, so, to avoid problems in passing from the Lie algebra to the group, we need to make some assumption on the fundamental group of G . (It needs to be a \mathbb{Q} -subgroup of the universal cover \tilde{G} .) Therefore, the statement of Proposition 6.1 requires G to be simply connected. Alternatively, one could require G to be adjoint, instead of simply connected, but the situation is not obvious for some intermediate groups that are neither adjoint nor simply connected.

Remark 6.2 As a complement to our explicit construction, it might be possible to use theorems of Galois cohomology to give a more elegant proof of Proposition 2.4. In this vein, G. Prasad (see [O, Proposition 6.4]) gave a very short proof of the existence of a \mathbb{Q} -form satisfying 6.1(1) and 6.1(3); perhaps a clever argument can yield 6.1(2) and/or 6.1(4), as well, but they do not seem to be obvious.

Let us set up the usual notation.

Notation 6.3

- \mathfrak{g} is a real semisimple Lie algebra;
- $\kappa(\cdot, \cdot)$ is the Killing form on \mathfrak{g} ;
- \mathfrak{h} is a maximal torus (i.e., a Cartan subalgebra) of \mathfrak{g} ;
- Φ is the set of roots of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$;
- h_{α} is the unique element of $\mathfrak{h}_{\mathbb{C}}$, such that $\alpha(t) = \kappa(t, h_{\alpha})$ for all $t \in \mathfrak{h}_{\mathbb{C}}$ (for each $\alpha \in \Phi$);
- $h_{\alpha}^* = 2h_{\alpha}/\kappa(h_{\alpha}, h_{\alpha})$ (for each $\alpha \in \Phi$);
- $(\mathfrak{g}_{\mathbb{C}})_{\alpha}$ is the root space corresponding to $\alpha \in \Phi$;
- θ is a Cartan involution of \mathfrak{g} , such that $\theta(\mathfrak{h}) = \mathfrak{h}$ (we also use θ to denote the extension to a \mathbb{C} -linear automorphism of $\mathfrak{g}_{\mathbb{C}}$);
- $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the Cartan decomposition of \mathfrak{g} corresponding to θ (i.e., \mathfrak{k} and \mathfrak{p} are the $+1$ and -1 eigenspaces of θ , respectively).

Because $\theta(\mathfrak{h}) = \mathfrak{h}$, we see that θ induces a permutation of Φ : we have $\theta((\mathfrak{g}_{\mathbb{C}})_{\alpha}) = (\mathfrak{g}_{\mathbb{C}})_{\theta(\alpha)}$.

The following lemma is a slight modification of a result of Borel [B, §3.2 and Lemma 3.5] that extends work of Chevalley and Weyl. (See [B, p. 116 and footnote on p. 117] for some historical remarks.) We follow Borel’s proof almost verbatim. However, Borel assumed that the Cartan subalgebra \mathfrak{h} contains a maximal \mathbb{R} -anisotropic torus of \mathfrak{g} , and, using this assumption, he obtained a stronger version of (3): $\theta(x_{\alpha}) = \pm x_{\theta(\alpha)}$.

Lemma 6.4 (Borel, Chevalley, Gantmacher, Weyl) *Assume the notation of (6.3).*

There is a function $\Phi \rightarrow \mathfrak{g}: \alpha \mapsto x_{\alpha}$, such that, for $\alpha, \beta \in \Phi$, we have

1. $x_{\alpha} \in \mathfrak{g}_{\alpha}$;
 2. $[x_{\alpha}, x_{\beta}] = \begin{cases} N_{\alpha, \beta} x_{\alpha + \beta} & \text{if } \alpha + \beta \in \Phi, \\ -h_{\alpha}^* & \text{if } \alpha + \beta = 0, \\ 0 & \text{if } 0 \neq \alpha + \beta \notin \Phi, \end{cases}$
- where
- $$N_{\alpha, \beta} = N_{-\alpha, -\beta} = \pm(p_{\alpha, \beta} + 1) \tag{6.1}$$

and $p_{\alpha, \beta} \geq 0$ is the greatest integer such that $\alpha - p_{\alpha, \beta}\beta \in \Phi$;

3. $\theta(x_{\alpha}) \in \{\pm x_{\theta(\alpha)}, \pm i x_{\theta(\alpha)}\}$; and
4. $\mathfrak{k} + i\mathfrak{p} = \sum_{\alpha \in \Phi} i\mathbb{R}h_{\alpha} + \sum_{\alpha \in \Phi} \{zx_{\alpha} + \bar{z}x_{-\alpha} \mid z \in \mathbb{C}\}$.

Proof [B, §3.2–§3.5] or [R1, Chap. 14] Fix a Chevalley basis

$$\{h_{\alpha}^* \mid \alpha \in \Delta\} \cup \{x_{\alpha} \mid \alpha \in \Phi\}$$

of \mathfrak{g} . Chevalley [C] showed that this satisfies (1) and (2).

Step 1. We may assume (4) holds. Recall that all of the maximal compact subgroups of any connected Lie group are conjugate to each other, and that all of the maximal toruses of any connected, compact Lie group are conjugate to each other. Thus, since the LHS and RHS of (4) are maximal compact subalgebras of $\mathfrak{g}_{\mathbb{C}}$ that contain the maximal torus $\sum_{\alpha \in \Phi} i\mathbb{R}h_{\alpha}$, they are conjugate, via an automorphism of $\mathfrak{g}_{\mathbb{C}}$ that normalizes $\sum_{\alpha \in \Phi} i\mathbb{R}h_{\alpha}$. Hence, by replacing $\{x_{\alpha}\}_{\alpha \in \Phi}$ with a conjugate, we may assume (4) holds.

Step 2. For each $\alpha \in \Phi$, define $c_{\alpha} \in \mathbb{C}$ by $\theta(x_{\alpha}) = c_{\alpha}x_{\theta(\alpha)}$; then

$$c_{-\alpha} = \frac{1}{c_{\alpha}} = c_{\theta(\alpha)} \text{ for all } \alpha \in \Phi, \tag{6.2}$$

and

$$c_\alpha c_\beta = \pm c_{\alpha+\beta} \text{ for all } \alpha, \beta \in \Phi, \text{ such that } \alpha + \beta \in \Phi. \quad (6.3)$$

Note that

$$\begin{aligned} -c_\alpha c_{-\alpha} h_{\theta(\alpha)}^* &= [c_\alpha x_{\theta(\alpha)}, c_{-\alpha} x_{-\theta(\alpha)}] \\ &= [\theta(x_\alpha), \theta(x_{-\alpha})] = \theta[x_\alpha, x_{-\alpha}] = -\theta(h_\alpha^*). \end{aligned}$$

Because $\theta(h_\alpha^*) = h_{\theta(\alpha)}^*$ (since θ is an automorphism that fixes \mathfrak{h}), this implies that $c_\alpha c_{-\alpha} = 1$, which establishes part of (6.2). For the other part, we use the fact that $\theta^2 = \text{Id}$ to calculate

$$x_\alpha = \theta^2(x_\alpha) = c_\alpha \theta(x_{\theta(\alpha)}) = c_\alpha c_{\theta(\alpha)} x_\alpha.$$

To establish (6.3), note that $p_{\theta(\alpha), \theta(\beta)} = p_{\alpha, \beta}$ (because θ is an automorphism), so $N_{\theta(\alpha), \theta(\beta)} = \pm N_{\alpha, \beta}$. Now use the fact that

$$[\theta(x_\alpha), \theta(x_\beta)] = \theta[x_\alpha, x_\beta].$$

Step 3. We may assume (3) holds. Let Δ be a basis of Φ (with respect to some order). Then Δ is a basis of the dual space \mathfrak{h}^* , so there is some $h \in \mathfrak{h}_\mathbb{C}$, such that $e^{\alpha(h)} = c_\alpha$, for every $\alpha \in \Delta$. Then, from (6.3), we see, by induction on the length of α , that $e^{\alpha(h)} = \pm c_\alpha$, for every $\alpha \in \Phi^+$. Because $c_{-\alpha} = 1/c_\alpha$, then we have $e^{\alpha(h)} = \pm c_\alpha$, for every $\alpha \in \Phi$.

For each $\alpha \in \Phi$, let $x'_\alpha = e^{-\alpha(h)/2} x_\alpha$. Then it is easy to see that (1) and (2) hold with x'_α in the place of x_α .

Because $\theta(\mathfrak{k} + i\mathfrak{p}) = \mathfrak{k} + i\mathfrak{p}$, we see from (4) that $c_{-\alpha} = \overline{c_\alpha}$. Then $1/c_\alpha = c_{-\alpha} = \overline{c_\alpha}$, so $|c_\alpha| = 1$. Therefore, $\alpha(h)$ is pure imaginary for every $\alpha \in \Delta$. By linearity, then $\alpha(h)$ is pure imaginary for every $\alpha \in \Phi$, so $e^{-\alpha(h)/2} = \overline{e^{\alpha(h)/2}}$, for every $\alpha \in \Phi$. Thus, (4) holds with x'_α in the place of x_α . For any $\alpha \in \Phi$, we have

$$\begin{aligned} \theta(x'_\alpha) &= e^{-\alpha(h)/2} \theta(x_\alpha) \\ &= (\pm c_\alpha)^{-1/2} (c_\alpha x_{\theta(\alpha)}) \\ &= (\pm c_\alpha)^{1/2} (e^{\theta(\alpha)(h)/2} x'_{\theta(\alpha)}) \\ &= (\pm c_\alpha)^{1/2} (\pm c_{\theta(\alpha)})^{1/2} x'_{\theta(\alpha)} \\ &= (\pm c_\alpha)^{1/2} (\pm c_\alpha)^{-1/2} x'_{\theta(\alpha)} \\ &= (\pm 1)^{1/2} x'_{\theta(\alpha)}. \end{aligned}$$

Thus, (3) holds with x'_α in the place of x_α .

□

Proposition 6.1 is obtained quite easily from this lemma. Most of the argument we give is based on [B] or [R1, Chap. 14]. However, Steps 6 and 7 are from [R2, §2], and Steps 8 and 9 are based on suggestions of G. Prasad (personal communication).

Proof of Proposition 6.1 We begin by establishing notation.

- Let \mathfrak{g} be the Lie algebra of G .
- Choose a maximal torus \mathfrak{h} of \mathfrak{g} , such that $\mathbb{R}\text{-rank } \mathfrak{h} = \mathbb{R}\text{-rank } \mathfrak{g}$.
- Assume the notation of (6.3).
- Let $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{p}$ (so \mathfrak{t} is a maximal \mathbb{R} -split torus of \mathfrak{g}).
- Let $\{x_\alpha\}_{\alpha \in \Phi}$ be as in Lemma 6.4.
- Let $\mathfrak{g}_{\mathbb{Q}(i)}$ be the $\mathbb{Q}(i)$ -span of $\{h_\alpha^*, x_\alpha\}_{\alpha \in \Phi}$ in $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$.
- Let $\mathfrak{h}_{\mathbb{Q}(i)} = \mathfrak{h}_{\mathbb{C}} \cap \mathfrak{g}_{\mathbb{Q}(i)}$ be the $\mathbb{Q}(i)$ -span of $\{h_\alpha^*\}_{\alpha \in \Phi}$ in $\mathfrak{g}_{\mathbb{C}}$.
- Let $\mathfrak{g}_{\mathbb{Q}} = \mathfrak{g}_{\mathbb{Q}(i)} \cap \mathfrak{g}$.

We will show that $\mathfrak{g}_{\mathbb{Q}}$ is a \mathbb{Q} -form of \mathfrak{g} , and that the corresponding \mathbb{Q} -form $G_{\mathbb{Q}}$ of G satisfies all the hypotheses of Proposition 6.1.

Step 1. $\mathfrak{g}_{\mathbb{Q}(i)}$ is a split $\mathbb{Q}(i)$ -form of $\mathfrak{g}_{\mathbb{C}}$, with $\mathfrak{h}_{\mathbb{Q}(i)}$ being a $\mathbb{Q}(i)$ -split maximal torus. It is clear that $\mathfrak{g}_{\mathbb{Q}(i)}$ is a $\mathbb{Q}(i)$ -form of $\mathfrak{g}_{\mathbb{C}}$ (because it is the $\mathbb{Q}(i)$ -span of a basis, and is closed under brackets). It is split because it contains a Chevalley basis of $\mathfrak{g}_{\mathbb{C}}$, and $\mathfrak{h}_{\mathbb{Q}(i)}$ is the maximal split torus corresponding to this basis.

Step 2. Each of \mathfrak{k} and \mathfrak{p} is the \mathbb{R} -span of its intersection with $\mathfrak{g}_{\mathbb{Q}}$. Because each of these subspaces is contained in \mathfrak{g} , it suffices to prove the conclusion with $\mathfrak{g}_{\mathbb{Q}}$ replaced by $\mathfrak{g}_{\mathbb{Q}(i)}$.

Let $u_{\mathbb{Q}} = (\mathfrak{k} + i\mathfrak{p}) \cap \mathfrak{g}_{\mathbb{Q}(i)}$. Then $u_{\mathbb{Q}}$ is a \mathbb{Q} -subspace of $\mathfrak{k} + i\mathfrak{p}$. Indeed, we see, from 6.4(4), that $u_{\mathbb{Q}}$ is a \mathbb{Q} -form of the real vector space $\mathfrak{k} + i\mathfrak{p}$.

Now $\mathfrak{k} + i\mathfrak{p}$ and $\mathfrak{g}_{\mathbb{Q}(i)}$ are θ -invariant (for the latter, see 6.4(3)). So $u_{\mathbb{Q}}$ is θ -invariant. This means that, with respect to the \mathbb{Q} -form $u_{\mathbb{Q}}$, the linear transformation $\theta|_{u_{\mathbb{Q}}}$ is defined over \mathbb{Q} . Since the eigenvalues (± 1) are rational, we conclude that the eigenspaces are spanned (over \mathbb{R}) by the rational vectors, that is by elements of $u_{\mathbb{Q}}$. Concretely, this means that the \mathbb{R} -span of $\mathfrak{k} \cap u_{\mathbb{Q}}$ is \mathfrak{k} , and the \mathbb{R} -span of $i\mathfrak{p} \cap u_{\mathbb{Q}}$ is $i\mathfrak{p}$. The first is exactly what we want to know about \mathfrak{k} . Multiplying by i transforms the second into exactly what we want to know about \mathfrak{p} .

Step 3. $\mathfrak{g}_{\mathbb{Q}}$ is a \mathbb{Q} -form of \mathfrak{g} . Because \mathfrak{g} and $\mathfrak{g}_{\mathbb{Q}(i)}$ are closed under brackets, it is clear that $\mathfrak{g}_{\mathbb{Q}}$ is a subalgebra of \mathfrak{g} . We just need to show that its \mathbb{R} -span is all of \mathfrak{g} .

From Step 2, we know that the \mathbb{R} -span of $\mathfrak{g}_{\mathbb{Q}}$ contains both \mathfrak{k} and \mathfrak{p} . Therefore, it contains $\mathfrak{k} + \mathfrak{p} = \mathfrak{g}$.

Step 4. $\mathfrak{g}_{\mathbb{Q}}$ splits over $\mathbb{Q}(i)$. We already pointed out in Step 1 that $\mathfrak{g}_{\mathbb{Q}(i)}$ is split.

Step 5. \mathbb{Q} -rank $\mathfrak{g}_{\mathbb{Q}} = \mathbb{R}$ -rank \mathfrak{g} . Because $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{p}$ is a maximal \mathbb{R} -split torus of \mathfrak{g} , it suffices to show that \mathfrak{t} is (defined over \mathbb{Q} and) \mathbb{Q} -split.

Substep 5.1. \mathfrak{t} is defined over \mathbb{Q} . From Step 3, we see that $\mathfrak{g}_{\mathbb{Q}(i)} = \mathfrak{g}_{\mathbb{Q}} + i\mathfrak{g}_{\mathbb{Q}}$, so, for any (real) subspace X of \mathfrak{g} , we have $X_{\mathbb{C}} \cap \mathfrak{g}_{\mathbb{Q}(i)} = (X \cap \mathfrak{g}_{\mathbb{Q}})_{\mathbb{C}}$. Thus, if $X_{\mathbb{C}}$ is the \mathbb{R} -span of its intersection with $\mathfrak{g}_{\mathbb{Q}(i)}$, then X is the \mathbb{R} -span of its intersection with $\mathfrak{g}_{\mathbb{Q}}$, i.e., X is defined over \mathbb{Q} .

It is clear, from the definition of $\mathfrak{g}_{\mathbb{Q}(i)}$, that $\mathfrak{h}_{\mathbb{C}}$ is the \mathbb{R} -span of its intersection with $\mathfrak{g}_{\mathbb{Q}(i)}$. Hence, from the preceding paragraph, we conclude that \mathfrak{h} is defined over \mathbb{Q} . From Step 2, we know that \mathfrak{p} is also defined over \mathbb{Q} . Hence, the intersection $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{p}$ is defined over \mathbb{Q} .

Substep 5.2. \mathfrak{t} is \mathbb{Q} -split. Let T be the \mathbb{Q} -torus of G corresponding to \mathfrak{t} . We know that T splits over \mathbb{R} , so $\chi(T_{\mathbb{R}}) \subset \mathbb{R}$, for every character χ of T . Because $\mathfrak{t} \subset \mathfrak{h}$, and \mathfrak{h} splits over $\mathbb{Q}(i)$ (see Step 1), we know that $\chi(T_{\mathbb{Q}}) \subset \mathbb{Q}(i)$, for every character χ of T . So $\chi(T_{\mathbb{Q}}) \subset \mathbb{R} \cap \mathbb{Q}(i) = \mathbb{Q}$, for every character χ of T ; hence, T is \mathbb{Q} -split.

Step 6. For each $\alpha \in \Phi$, let

$$\mathfrak{g}_{\pm\alpha} = \langle (\mathfrak{g}_{\mathbb{C}})_{\alpha}, (\mathfrak{g}_{\mathbb{C}})_{-\alpha} \rangle \cap \mathfrak{g}.$$

If $\alpha(\mathfrak{t}) = 0$, then $(\mathfrak{g}_{\pm\alpha})_{\mathbb{Q}} \cong \mathfrak{su}(2)_{\mathbb{Q}}$ (for the usual \mathbb{Q} -form on $SU(2)$). Because \mathfrak{t} is a maximal \mathbb{R} -split torus, the assumption on α implies that $(\mathfrak{g}_{\mathbb{C}})_{\alpha} \subset \mathfrak{k}_{\mathbb{C}}$ (and the same for $-\alpha$). So, using 6.4(4), we see that

$$\begin{aligned} \langle (\mathfrak{g}_{\mathbb{C}})_{\alpha}, (\mathfrak{g}_{\mathbb{C}})_{-\alpha} \rangle \cap \mathfrak{g} &= \langle (\mathfrak{g}_{\mathbb{C}})_{\alpha}, (\mathfrak{g}_{\mathbb{C}})_{-\alpha} \rangle \cap \mathfrak{k} \\ &= i\mathbb{R}h_{\alpha}^* + \{zx_{\alpha} + \bar{z}x_{-\alpha} \mid z \in \mathbb{C}\}. \end{aligned}$$

Therefore

$$\begin{aligned} \langle \langle (\mathfrak{g}_{\mathbb{C}})_{\alpha}, (\mathfrak{g}_{\mathbb{C}})_{-\alpha} \rangle \cap \mathfrak{g} \rangle_{\mathbb{Q}} &= i\mathbb{Q}h_{\alpha}^* + \{zx_{\alpha} + \bar{z}x_{-\alpha} \mid z \in \mathbb{Q}(i)\} \\ &\cong \mathfrak{su}(2)_{\mathbb{Q}}. \end{aligned}$$

Step 7. Every element of the Weyl group of the anisotropic kernel of G is realized over \mathbb{Q} . The Weyl element of $SU(2)$ is realized by the rational matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

So Step 6 implies that all of the root reflections of the anisotropic kernel can be realized over \mathbb{Q} . These reflections generate the entire Weyl group.

Step 8. For any odd prime p , and any $\alpha \in \Phi$, such that $\alpha(t) = 0$, the Lie algebra $\mathfrak{g}_{\pm\alpha}$ is \mathbb{Q}_p -split. The quadratic form $x_1^2 + x_2^2 + x_3^2$ is isotropic over \mathbb{Q}_p (see, for example, [BS, Corollary 1.6.2, p. 50]), so $\mathfrak{so}(3)$ is \mathbb{Q}_p -split. Since $\mathfrak{so}(3)_{\mathbb{Q}_p} \cong \mathfrak{su}(2)_{\mathbb{Q}_p}$ and $\mathfrak{su}(2)_{\mathbb{Q}_p} \cong (\mathfrak{g}_{\pm\alpha})_{\mathbb{Q}_p}$ (see Step 6), we conclude that $\mathfrak{g}_{\pm\alpha}$ is \mathbb{Q}_p -split.

Step 9. $\mathfrak{g}_{\mathbb{Q}_p}$ is quasi-split, for every odd prime p . Let Ψ be a maximal set of pairwise orthogonal roots in $\{\alpha \in \Phi \mid \alpha(t) = 0\}$. For each $\alpha \in \Psi$, we know, from Step 8, that $(\mathfrak{g}_{\pm\alpha})_{\mathbb{Q}_p}$ contains a nontrivial \mathbb{Q}_p -split torus \mathfrak{s}_α ; let

$$\mathfrak{s}_{\mathbb{Q}_p} = \mathfrak{t}_{\mathbb{Q}_p} + \sum_{\alpha \in \Psi} \mathfrak{s}_\alpha.$$

From the maximality of Ψ , we know that the centralizer of the torus

$$\mathfrak{s}' = \mathfrak{t}_{\mathbb{Q}_p} + \sum_{\alpha \in \Psi} i\mathbb{Q}_p h_\alpha^*$$

in $\mathfrak{g}_{\mathbb{Q}_p}$ is $\mathfrak{h}_{\mathbb{Q}_p}$, a (maximal) torus of $\mathfrak{g}_{\mathbb{Q}_p}$. Now both \mathfrak{s} and \mathfrak{s}' are maximal tori of the Lie algebra

$$\mathfrak{t}_{\mathbb{Q}_p} + \sum_{\alpha \in \Psi} \mathfrak{g}_{\pm\alpha},$$

so they are conjugate over the algebraic closure of \mathbb{Q}_p . Therefore, the centralizer of \mathfrak{s} is also a (maximal) torus of $\mathfrak{g}_{\mathbb{Q}_p}$. Because \mathfrak{s} is \mathbb{Q}_p -split, this implies that $\mathfrak{g}_{\mathbb{Q}_p}$ is quasi-split. □

7 Which \mathbb{Q} -forms are \mathbb{R} -universal?

Definition 7.1 Let us say that a Lie algebra $\mathfrak{g}_{\mathbb{Q}}$ over \mathbb{Q} is *universal for real representations* (or simply *\mathbb{R} -universal*, for short) if every real representation of $\mathfrak{g}_{\mathbb{Q}}$ has a \mathbb{Q} -form.

We have shown that every semisimple real Lie algebra has an \mathbb{R} -universal \mathbb{Q} -form (see 2.6). Furthermore, our construction yields an example that is $\mathbb{Q}(i)$ -split (see 6.1(1)). In fact, results of J. Tits [T] show it is often the case that every $\mathbb{Q}(i)$ -split \mathbb{Q} -form of \mathfrak{g} is \mathbb{R} -universal.

Proposition 7.2 *Let \mathfrak{g} be a compact, simple Lie algebra over \mathbb{R} . If \mathfrak{g} has a \mathbb{Q} -form that splits over some quadratic extension of \mathbb{Q} , but is not \mathbb{R} -universal, then either*

$$\mathfrak{g} \cong \mathfrak{su}(n), \text{ for some even } n \geq 4,$$

or

$$\mathfrak{g} \cong \mathfrak{so}(n), \text{ for some } n \not\equiv 3, 5 \pmod{8}.$$

In this section, we show that the converse is true (cf. 7.4).

Using Tits' approach, it should be possible to give an explicit list of the \mathbb{Q} -forms of $\mathfrak{su}(n)$ and $\mathfrak{so}(n)$ that are not \mathbb{R} -universal (except, perhaps, those that are triality forms of type 3D_4 or 6D_4). The author intends to attack this project in a future paper.

Proof of Proposition 7.2 Let F be a quadratic extension of \mathbb{Q} , and suppose \mathfrak{g}_F is a \mathbb{Q} -form of \mathfrak{g} that splits over F , but is not \mathbb{R} -universal. (We remark that F must be an imaginary extension, because $\mathfrak{g}_{\mathbb{R}} = \mathfrak{g}$ is compact, not split.) There is an irreducible \mathbb{Q} -representation V of \mathfrak{g}_F , such that V is reducible over \mathbb{R} (see 3.3). Thus, we may write $V_{\mathbb{R}} = W \oplus X$ (with W and X nontrivial). From Corollary 3.2(2), we see that $W_{\mathbb{C}}$ and $X_{\mathbb{C}}$ are irreducible.

Let λ be the highest weight of $W_{\mathbb{C}}$, and let w be the longest element of the Weyl group of \mathfrak{g} . Because $W_{\mathbb{C}}$ is irreducible, and has an \mathbb{R} -form (namely, W), it must be the case that

- $w(\lambda) = -\lambda$, and
- when λ is expressed as a linear combination of simple roots, the sum of the coefficients is an integer

(see [T, Proposition 6.1 and following comments]). Because $W_{\mathbb{C}}$ has no \mathbb{Q} -form (any such \mathbb{Q} -form would be isomorphic to a proper submodule of V), it must be the case that

- λ is not an integral linear combination of roots

(see [T, Theorem 3.3]). Now Lemma 7.3 below yields the desired conclusion.

□

The following observation is obtained by inspection of a list of the fundamental dominant weights of the complex simple Lie algebras of each type $A_\ell, B_\ell, C_\ell, D_\ell, E_6, E_7, E_8, F_4, G_2$. Such a list appears in [H, Table 1, p. 69].

Lemma 7.3 *Suppose \mathfrak{g} is a simple Lie algebra over \mathbb{C} , and let w be the longest element of the Weyl group of \mathfrak{g} . There is a dominant weight λ of \mathfrak{g} , such that*

1. $w(\lambda) = -\lambda$, and
2. when λ is expressed as a linear combination of simple roots, the sum of the coefficients is an integer, and
3. λ is not an integral linear combination of roots,

if and only if either

- a. \mathfrak{g} is of type A_ℓ , with ℓ odd (and $\ell \geq 3$), or
- b. \mathfrak{g} is of type B_ℓ , with $\ell \equiv 3$ or $4 \pmod{4}$ (and $\ell \geq 3$), or
- c. \mathfrak{g} is of type D_ℓ (and $\ell \geq 3$).

Combining Lemma 7.3 with the following result establishes the converse of Proposition 7.2.

Proposition 7.4 *Suppose that G is a compact, real, semisimple Lie group, \mathfrak{g} is the Lie algebra of G , \mathfrak{t} is a maximal torus of \mathfrak{g} , and Φ is the root system of \mathfrak{g} (with respect to \mathfrak{t}). Let $w \in G$ be a representative of the longest element of the Weyl group of G . Let V be an irreducible \mathbb{C} -representation of G and λ be the highest weight of V . If*

- $w(\lambda) = -\lambda$;
- when λ is expressed as a linear combination of simple roots, the sum of the coefficients is an integer; and
- $\lambda \notin \langle \Phi \rangle$ (that is, λ is not an integral linear combination of roots);

then there exist

1. a real form $V_{\mathbb{R}}$ of V ; and
2. a \mathbb{Q} -form $\mathfrak{g}'_{\mathbb{Q}}$ of \mathfrak{g} ;

such that

- a. $\mathfrak{g}'_{\mathbb{Q}}$ splits over $\mathbb{Q}(i)$; and
- b. $V_{\mathbb{R}}$ does not have a \mathbb{Q} -form (with respect to $\mathfrak{g}'_{\mathbb{Q}}$).

Proof (1) Since $w(\lambda) = -\lambda$, we see that the lowest weight of V is $-\lambda$. Since the dual of V is the irreducible \mathfrak{g} -module whose lowest weight is $-\lambda$, we conclude that V is self-dual. That is, V is isomorphic to its dual.

Because \mathfrak{g} is compact, there is a \mathfrak{g} -invariant Hermitian form on V , so V is conjugate-isomorphic to its dual. Combining this with the conclusion

of preceding paragraph, we conclude that V is conjugate-isomorphic to itself. That is, V is isomorphic to its conjugate. Therefore $V \otimes_{\mathbb{R}} \mathbb{C} \cong V \oplus V$ is the direct sum of two isomorphic irreducibles.

Because the sum of the coefficients of λ is an integer, V has a real form $V_{\mathbb{R}}$ (see 7.6). (We remark that any two real forms of V are isomorphic, so it does not matter which one is chosen.)

(2a) Let Δ be a base of Φ . The difference of the highest weight and the lowest weight is a sum of roots, so, because the highest weight is λ and the lowest weight is $-\lambda$, we may write

$$2\lambda = \sum_{\delta \in \Delta}^r a_{\delta} \delta,$$

with each $a_{\delta} \in \mathbb{Z}$. Because $\lambda \notin \langle \Phi \rangle$, there must be some $\tau \in \Delta$, such that a_{τ} is odd.

Let

$$\{h_{\delta}\}_{\delta \in \Delta} \cup \{x_{\alpha}\}_{\alpha \in \Phi}$$

be the usual Chevalley basis of $\mathfrak{g}_{\mathbb{C}}$, so that

$$\mathfrak{g} = \bigoplus_{\delta \in \Delta} i\mathbb{R}h_{\delta} \oplus \bigoplus_{\alpha \in \Phi} \{zx_{\alpha} + \bar{z}x_{-\alpha} \mid z \in \mathbb{C}\}.$$

For each $\alpha \in \Phi$, we may write

$$\alpha = \sum_{\delta \in \Delta} c_{\delta}(\alpha) \delta,$$

with each $c_{\delta}(\alpha) \in \mathbb{Z}$. In particular, we have defined a function $c_{\tau} : \Phi \rightarrow \mathbb{Z}$. Now let

$$\mathfrak{g}'_{\mathbb{Q}(i)} = \bigoplus_{\delta \in \Delta} \mathbb{Q}(i)h_{\delta} \oplus \bigoplus_{\alpha \in \Phi} (\sqrt{3})^{c_{\tau}(\alpha)} (\mathbb{Q}(i)x_{\alpha} + \mathbb{Q}(i)x_{-\alpha})$$

and

$$\begin{aligned} \mathfrak{g}'_{\mathbb{Q}} &= \mathfrak{g}'_{\mathbb{Q}(i)} \cap \mathfrak{g} \\ &= \bigoplus_{\delta \in \Delta} i\mathbb{Q}h_{\delta} \oplus \bigoplus_{\alpha \in \Phi} (\sqrt{3})^{c_{\tau}(\alpha)} \{zx_{\alpha} + \bar{z}x_{-\alpha} \mid z \in \mathbb{Q}(i)\}. \end{aligned}$$

Because

$$\mathbb{Q}(i)(\sqrt{3})^{-r} = 3^r \mathbb{Q}(i)(\sqrt{3})^{-r} = \mathbb{Q}(i)(\sqrt{3})^r,$$

we see that $\mathfrak{g}'_{\mathbb{Q}(i)}$ is the $\mathbb{Q}(i)$ -span of the Chevalley basis

$$\{h_{\delta}\}_{\delta \in \Delta} \cup \left\{ (\sqrt{3})^{c_{\tau}(\alpha)} x_{\alpha} \right\}_{\alpha \in \Phi},$$

so $\mathfrak{g}'_{\mathbb{Q}(i)}$ is a split $\mathbb{Q}(i)$ -form of $\mathfrak{g}_{\mathbb{C}}$. From this (and because it is obvious that the \mathbb{R} -span of $\mathfrak{g}'_{\mathbb{Q}}$ is all of \mathfrak{g}), it follows that $\mathfrak{g}'_{\mathbb{Q}}$ is a \mathbb{Q} -form of \mathfrak{g} , such that $\mathfrak{g}'_{\mathbb{Q}}$ splits over $\mathbb{Q}(i)$.

(2b) Suppose $V_{\mathbb{R}}$ has a $\mathfrak{g}'_{\mathbb{Q}}$ -invariant \mathbb{Q} -form $V'_{\mathbb{Q}}$. (This will lead to a contradiction.) Then there is a $\mathfrak{g}'_{\mathbb{Q}}$ -equivariant conjugate-linear involution $\sigma': V'_{\mathbb{Q}(i)} \rightarrow V'_{\mathbb{Q}(i)}$.

Let $\mathfrak{g}_{\mathbb{Q}}$ be the standard \mathbb{Q} -form of \mathfrak{g} (obtained by replacing $\sqrt{3}$ with 1 in the above construction). We know (from Corollary 2.3) that $V_{\mathbb{R}}$ has a $\mathfrak{g}_{\mathbb{Q}}$ -invariant \mathbb{Q} -form $V_{\mathbb{Q}}$. Let $\sigma: V_{\mathbb{Q}(i)} \rightarrow V_{\mathbb{Q}(i)}$ be the corresponding $\mathfrak{g}_{\mathbb{Q}}$ -equivariant conjugate-linear involution.

Because all highest-weight vectors of V are scalar multiples of each other, there is no harm in assuming that $(V'_{\mathbb{Q}(i)})^{\lambda} \cap V_{\mathbb{Q}(i)}^{\lambda} \neq 0$. (Simply replace $V_{\mathbb{Q}(i)}$ with $kV_{\mathbb{Q}(i)}$, for some $k \in \mathbb{C}$.) Then, because these are one-dimensional vector spaces over $\mathbb{Q}(i)$, we must have

$$(V'_{\mathbb{Q}(i)})^{\lambda} = V_{\mathbb{Q}(i)}^{\lambda}.$$

Then, by induction on the length of α , it is easy to see that if α is any integral combination of elements of Δ , with non-negative coefficients, then

$$(V'_{\mathbb{Q}(i)})^{\lambda-\alpha} = (\sqrt{3})^{c_{\tau}(\alpha)} V_{\mathbb{Q}(i)}^{\lambda-\alpha}.$$

In particular, because $c_{\tau}(2\lambda)$ is odd (this was how τ was chosen), we know that

$$(V'_{\mathbb{Q}(i)})^{-\lambda} = \sqrt{3} V_{\mathbb{Q}(i)}^{-\lambda}.$$

Because $\lambda(\mathfrak{h}'_{\mathbb{Q}}) = \lambda(\mathfrak{h}_{\mathbb{Q}}) \subset i\mathbb{Q}$, we have

$$\sigma(V_{\mathbb{Q}(i)}^{-\lambda}) = V_{\mathbb{Q}(i)}^{\lambda} \text{ and } \sigma'((V'_{\mathbb{Q}(i)})^{\lambda}) = (V'_{\mathbb{Q}(i)})^{-\lambda}.$$

Let $f = \sigma' \circ \sigma: V \rightarrow V$, so f is \mathbb{C} -linear and \mathfrak{g} -equivariant; hence, f is a scalar, say $f(v) = kv$. Therefore

$$\begin{aligned} k \cdot V_{\mathbb{Q}(i)}^{-\lambda} &= f(V_{\mathbb{Q}(i)}^{-\lambda}) = \sigma'(\sigma(V_{\mathbb{Q}(i)}^{-\lambda})) = \sigma'(V_{\mathbb{Q}(i)}^{\lambda}) \\ &= \sigma'((V'_{\mathbb{Q}(i)})^{\lambda}) = (V'_{\mathbb{Q}(i)})^{-\lambda} = \sqrt{3} V_{\mathbb{Q}(i)}^{-\lambda}, \end{aligned}$$

so $k = \sqrt{3}k'$, for some $k' \in \mathbb{Q}(i)$.

We have

$$(\sigma')^2 = (f \circ \sigma)^2 = (k\sigma)^2 = (\sqrt{3}k'\sigma)^2 = 3k'\overline{k'} \neq 1$$

(because 3 is not a sum of two rational squares). This contradicts the fact that σ' is an involution.

□

Example 7.5 The direct sum of \mathbb{R} -universal Lie algebras need not be \mathbb{R} -universal. For example, let $\mathfrak{g}_1 = \mathfrak{so}(5)$ and $\mathfrak{g}_2 = \mathfrak{so}(11)$. For $j = 1, 2$, the Lie algebra $(\mathfrak{g}_j)_{\mathbb{C}}$ has a (fundamental) dominant weight λ_j , such that, when λ_j is expressed as a linear combination of simple roots, the sum of the coefficients is a half-integer. Then the weight $\lambda_1 \otimes \lambda_2$ of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ satisfies the hypotheses of Proposition 7.4, so $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ has a \mathbb{Q} -form that is not \mathbb{R} -universal. However, any \mathbb{Q} -form of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ must be the direct sum of \mathbb{Q} -forms of the factors and those, by Proposition 7.2, are \mathbb{R} -universal.

The following useful observations are well known. (4a \Leftrightarrow 4b) follows from Schur's Lemma. (4b \Leftrightarrow 4c) follows from [St, Lemma 79(b), p. 226]. (4b \Leftrightarrow 4d) follows from [T, Proposition 6.1]. (4a \Leftrightarrow 4c) is implicit in the proof on pp. 137–138 of [R2].

Lemma 7.6 *Suppose G is a compact, semisimple, real Lie group, V is an irreducible, self-dual, real representation of G and λ is the highest weight of V . Let $w \in G$ be a representative of the longest element of the Weyl group of G . Then*

1. We have $\text{End}_G(V) \cong \mathbb{R}$ or \mathbb{H} .
2. We have $w(\lambda) = -\lambda$ and $\lambda(w^2) = \pm 1$.
3. We may write λ as a linear combination of the fundamental dominant weights of G , and the sum s of the coefficients in this linear combination is either an integer or a half-integer.
4. The following are equivalent:
 - (a) $V \otimes_{\mathbb{R}} \mathbb{C}$ is irreducible;
 - (b) $\text{End}_G(V) \cong \mathbb{R}$;
 - (c) $\lambda(w^2) = 1$;
 - (d) the sum s is an integer.

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DEPARTMENT OF MATHEMATICS, OKLAHOMA STATE UNIVERSITY,
STILLWATER, OK 74078, U.S.A.

E-mail: dwitte@math.okstate.edu

URL: <http://www.math.okstate.edu/~dwitte>

Current Address: DEPARTMENT OF MATHEMATICS AND COMPUTER
SCIENCE, UNIVERSITY OF LETHBRIDGE, LETHBRIDGE, ALBERTA, T1K
3M4, CANADA

E-mail: Dave.Morris@uleth.ca

URL: <http://people.uleth.ca/~dave.morris>