Creating Repeating Hyperbolic Patterns

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ABSTRACT

A process for creating repeating patterns of the hyperbolic plane is described. Unlike the Euclidean plane, the hyperbolic plane has infinitely many different kinds of repeating patterns. The Poincare circle model of hyperbolic geometry has been used by the artist M. C. Escher to display interlocking, repeating, hyperbolic patterns. A program has been designed which will do this automatically. The user enters a motif, or basic subpattern, which could theoretically be replicated to fill the hyperbolic plane. In practice, the replication process can be iterated sufficiently often to appear to fill the circle model. There is an interactive "boundary procedure" which allows the user to design a motif which will be replicated into a completely interlocking pattern. Duplication of two of Escher's patterns and some entirely new patterns are included in the paper.

KEY WORDS AND PHRASES: hyperbolic geometry, Poincare circle model, tessellations, symmetry groups, interactive graphics, motif, repeating pattern, M. C. Escher, computer art.
CR CATEGORIES: 3.15, 3.41, 8.2

1. INTRODUCTION

This paper describes a process by which repeating patterns of the hyperbolic plane may be generated. A repeating pattern is defined to be a pattern which remains invariant under certain transformations of the hyperbolic plane. The Poincare circle model of hyperbolic geometry gives a concrete realization of the hyperbolic plane [Coxeter, 1961]. The points of this model are the interior points of a circle, called the bounding circle; the hyperbolic lines are represented by the diameters of the bounding circle and circular arcs orthogonal to the bounding circle.

The hyperbolic transformations of most interest to us are reflections across hyperbolic lines and rotations about points. Hyperbolic reflections consist of ordinary Euclidean reflections across diameters and inversions with respect to the circular arcs. A rotation about a point can be produced by successive reflections across two lines intersecting at that point and at an angle equal to half the angle of rotation.

Given a repeating pattern, the symmetry group of the pattern consists of all the transformations which preserve that pattern. Conversely, given a group of transformations and a basic subpattern or motif, a repeating pattern can be constructed by parqueting the hyperbolic plane with copies of the motif obtained by applying transformations from the group to the original motif. This is the process that will be described in this paper.

The user first selects one of the four kinds of groups of transformations. These groups will be described is Section 2. Then a motif is entered interactively with the aid of a "boundary procedure", to be described in Section 4. The motifs can be designed to form an interlocking pattern, if desired. Finally, the copies of the motif are replicated about the circle model of the hyperbolic plane, using transformations from the selected group.

2. THE SYMMETRY GROUPS

There are infinitely many types of repeating patterns of the hyperbolic plane, giving rise to infinitely many symmetry groups. This paper will concentrate on four families of symmetry groups whose patterns are highly symmetric.

A regular tessellation, {p,q}, of the hyperbolic plane is a covering of the hyperbolic plane by regular p-sided polygons, or simply p-gons, meeting only edge-to-edge and vertex-to-vertex, q at a vertex [see Coxeter and Moser, 1957 for notation]. It is necessary that $(p-2)*(q-2) > 4$ in order that the q vertex angles add to 360 degrees. The heavy lines of Figure 1 show a {6,4}.

The symmetry group of the tessellation {p,q} consists of reflections across three kinds of lines: the edges of the p-gons, the perpendicular bisectors of the edges, and the radii of the p-gons which pass through the vertices. This symmetry group is denoted [p,q] [Coxeter, Moser, 1957]. The light and heavy lines of Figure 1 show the lines of reflective symmetry of the {6,4}. These lines of reflective symmetry divide the hyperbolic plane into congruent hyperbolic right triangles with acute angles of π/p and π/q. Any one of these triangles is a fundamental region for the group [p,q] since the images of one such triangle by all the transformations in [p,q] will exactly cover the hyperbolic plane. Thus to create a repeating pattern with this symmetry group, it is necessary only to create a motif within one of the triangles and then successively reflect that motif over the hyperbolic plane. Note that the motif need not fill the fundamental region. Also the reflecting edges form a natural boundary beyond which the motif need not be extended, since the pattern will automatically be extended there by the reflection process. Figure 2 shows a motif within a fundamental region for the {6,4} and Figure 3 shows the complete pattern generated by the motif.

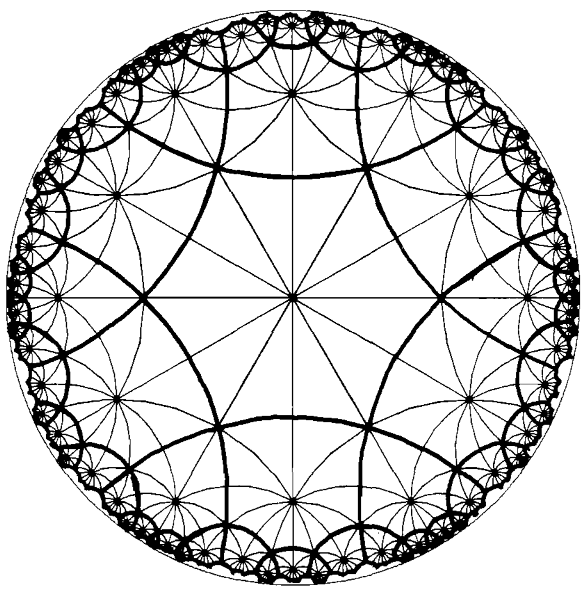


Figure 1.

The tessellation {6,4} (outlined in dark lines), showing all lines of reflective symmetry (both dark and light lines).

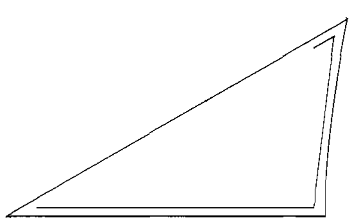


Figure 2.

A triangular fundamental region for the group [6,4] containing a bent arrow motif.

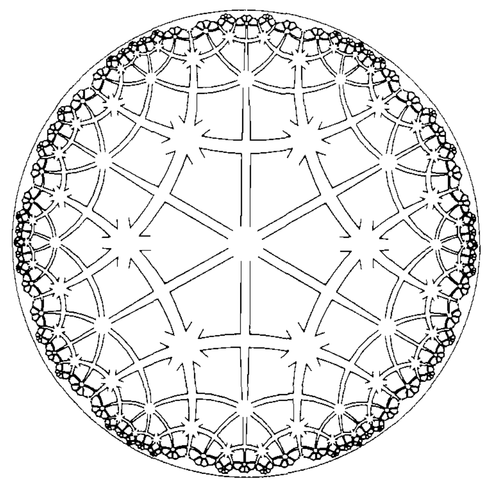


Figure 3.

The repeating pattern generated by the motif of Figure 2 and the symmetry group [6,4].

The tessellation {p,q} also has rotational symmetries of orders p, q, and 2 about the centers, vertices, and centers of the edges, respectively, of the p-gons. We denote this (orientation preserving) group of rotations by [p,q]+. A fundamental region in this case can be taken to be an isosceles triangle with angles $2\pi/p$, π/q, and π/q formed from two of the triangles in the previous case (Figure 4).

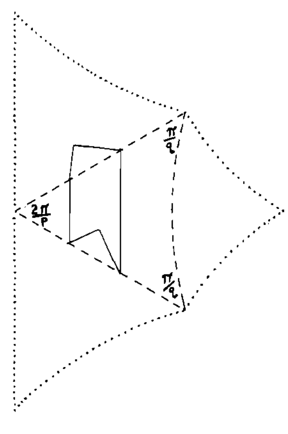


Figure 4.

The dashed lines outline a fundamental region for the group [6,4]+. The dotted lines outline adjacent copies of the fundamental region.

In this case, however, there are no natural boundaries. It is up to the user to supply the motif boundary. The motif boundary can extend over the edge of the triangular fundamental region, provided that it is correspondingly indented elsewhere along the edge of the fundamental region (Figure 4). An interactive "boundary procedure", described in Section 4, aids the user in this process. If the motif and any corresponding indentations fill the triangular fundamental region, then the motif itself can be taken to be the new fundamental region. In this case, the pattern formed by the transformed images of the motif will be completely interlocking.

If p-fold rotational symmetry about the centers of the p-gons of a {p,q} tessellation is combined with reflective symmetry across the edges, then a new group of symmetries, [p+,q], is obtained. Here, q must be even so that reflective symmetry occurs only across edges. The fundamental region can be taken to be the same isosceles triangle used for the previous group. The base of the isosceles triangle, being a line of reflection, forms a natural boundary, but the interactive motif "boundary procedure" is needed for the other two sides (Figure 5a).

On the other hand, the symmetry group consisting of q-fold rotational symmetries about the vertices of a tessellation {p,q} together with reflective symmetries across the perpendicular bisectors of the edges is denoted by [p,q+]. In this case p must be even. The fundamental region can be taken to be a kite-shaped area formed by joining two of the right triangles of the [p,q] case along a hypotenuse (Figure 5b).

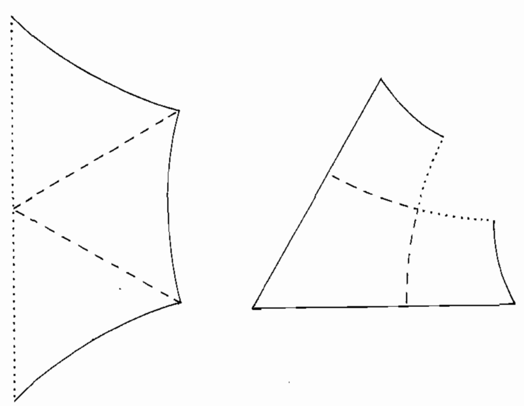


Figure 5a. Fundamental region for the group [p+,q]
Figure 5b. Fundamental region for the group [p,q+].

Solid lines indicate lines of reflective symmetry. Dashed lines complete the outline of the fundamental region. Dotted lines complete the outlines of adjacent copies of the fundamental region.

The two edges of the kite corresponding to reflection lines form a natural boundary, but again, the interactive "boundary procedure" is needed for the other two sides. (From an abstract point of view, this group is the same as the previous one. It is distinguished from that case by having reflective rather than rotational symmetry about the center of the Poincare circle model.)

3. HISTORY

The first well-known repeating patterns of the hyperbolic plane were the tessellations {p,q} which appeared in mathematical expositions [Fricke and Klein, 1890]. Often, for clarity, one half of each of the isosceles triangular fundamental regions for the group [p,q]+ were shaded, the other half being left blank. Figure 6 shows one such pattern with symmetry group [6,4]+.

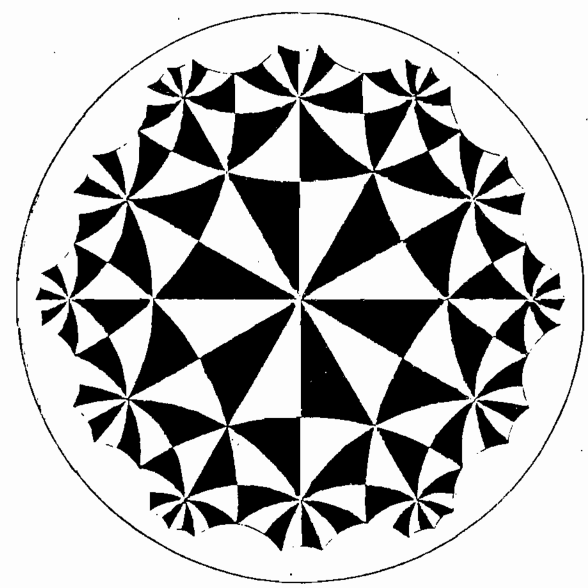


Figure 6. A pattern with symmetry group [6,4]+.

This pattern appeared in an article by H. S. M. Coxeter (Coxeter, 1957) which inspired the Dutch artist M. C. Escher to create more complicated repeating patterns of interlocking motifs. Two of the four hyperbolic patterns which he created are shown in Figures 7 and 8, with symmetry groups [8,3+] (if differences in shading are ignored) and [6,4+] respectively. In 1978, Alexander (Alexander, 1978) developed a computer program to generate repeating patterns with symmetry group [p,q] once the coordinates of the motif had been entered.

Figure 7.

M. C. Escher's print Circle Limit II, taken from "The World of M. C. Escher" [Locher 1971]. If shading is ignored, this pattern has symmetry group [8,3+].

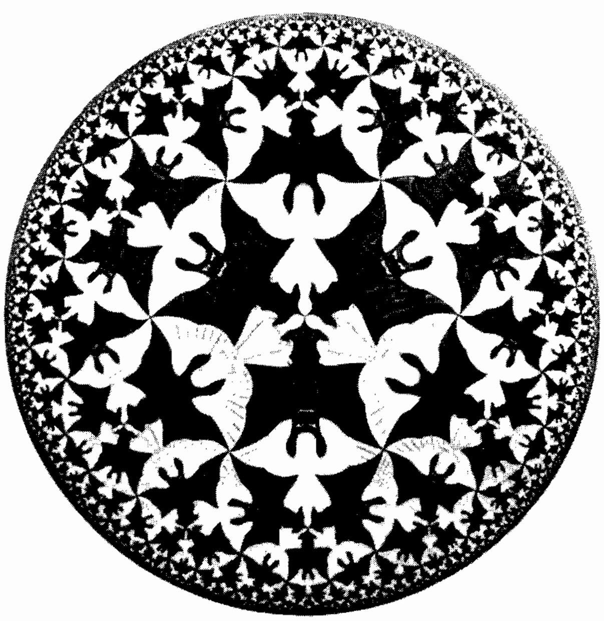


Figure 8.

M. C. Escher's print Circle LImit IV, taken from "The World of M. C. Escher" [Locher 1971]. The symmetry group of this pattern is [6,4+].

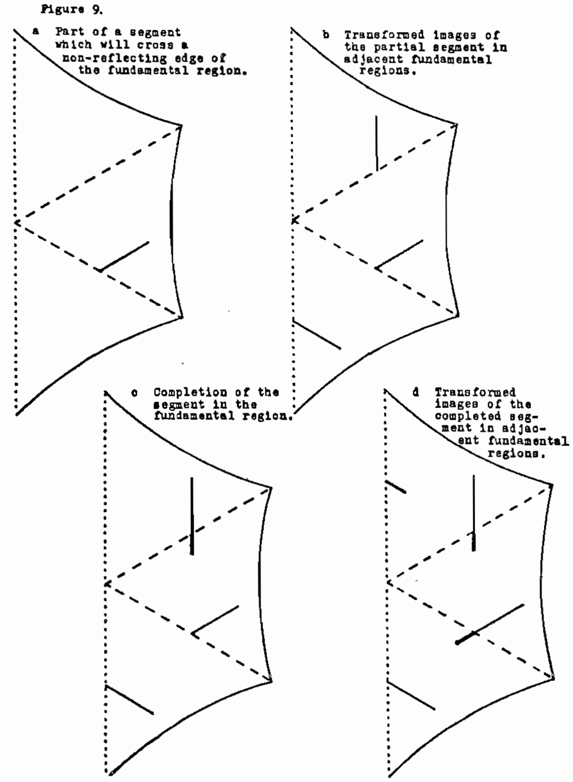
4. THE PATTERN CREATION PROCESS

The process begins with the choice of one of the four groups [p,q], [p,q]+, [p+,q], or [p,q+] which will be the symmetry group of the final pattern. Once the group has been chosen, the corresponding fundamental region is displayed on the graphics screen. The natural boundaries (i. e. lines of reflective symmetry) of the fundamental region are drawn as solid lines; the other edges (where the interactive "boundary procedure" applies) are drawn as dashed lines. Copies of the fundamental regions which are adjacent to the original fundamental region across non-reflecting edges are outlined with solid lines (corresponding to other reflection lines) and dotted lines (Figures 2, 4, and 5). There are zero, three, two, and two of these adjacent copies of the fundamental region corresponding to the groups [p,q], [p,q]+, [p+,q], and [p,q+] respectively.

The second step is the creation of the motif within the fundamental region. In the case of the group [p,q], this is a straightforward process of moving and drawing, using a cursor (since the fundamental region has natural boundaries).

The second step for the other groups is more interesting, since it is possible to draw line segments across the non-reflecting edges of the fundamental region. The interactive motif boundary procedure is required to do this. It works as follows: First, that part of the segment between the present position and the edge is drawn (Figure 9a). The boundary procedure then draws the transformed image of that partial segment in each of the adjacent copies of the fundamental region (Figure 9b). Finally it is necessary to move the

Figure 9.



cursor to the endpoint of the transformed image (which will also be on a non-reflecting edge of the fundamental region) and then draw the remainder of the original line segment (Figures 9c and 9d).

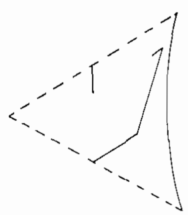
In fact, the interactive boundary procedure draws the transformed images of all segments, since it is up to the user to decide which line segments will eventually form the motif boundary. As mentioned before, if the motif and its transformed images fill the fundamental region, then a completely interlocking pattern, like those of M. C. Escher, will be created.

During the second step, the moves, draws, and color changes are stored in three arrays: Action, which records the action taken, and X and Y which record the (terminal) location of the action (terminal location = initial location for a color change).

The final step, replication, will be described in the next section.

5. THE REPLICATION ALGORITHM

The replication process occurs in two stages. The first stage is described as follows: For simplicity, the left-most vertex of the fundamental region is taken to be the center of the bounding circle. The fundamental region may be successively reflected and/or rotated about this vertex to form a complete p-gon, the central p-gon, centered within the bounding circle. The corresponding collection of images of the motif produced by these reflections and rotations is called the p-gon pattern (Figure 10).



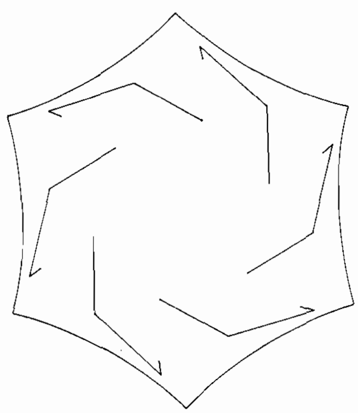


Figure 10a. A motif within the fundamental region for the group [6+,4].
Figure 10b. Replication of the motif of Figure 10a to fill the central p-gon, giving the p-gon pattern.

The arrays X, Y, and Action are extended to record the entire p-gon pattern. This is done, for each of the refelections or rotations, as follows: if n actions were required to create the motif, then Action[i+n] is the same as Action[i], and X[i+n] and Y[i+n] are computed by applying the reflection or rotation to the point (X[i], Y[i]). (Note that these reflections and rotations are ordinary Euclidean reflections and rotations since they are performed about the center of the bounding circle.)

In order to describe the second stage of the replication process, we define p-gon layers inductively as follow: the 0-th layer contains only the central p-gon; the (k+1)-st layer contains all those p-gons not in any previous layer but which share an edge or vertex with a p-gon from the k-th layer (Figure 11).

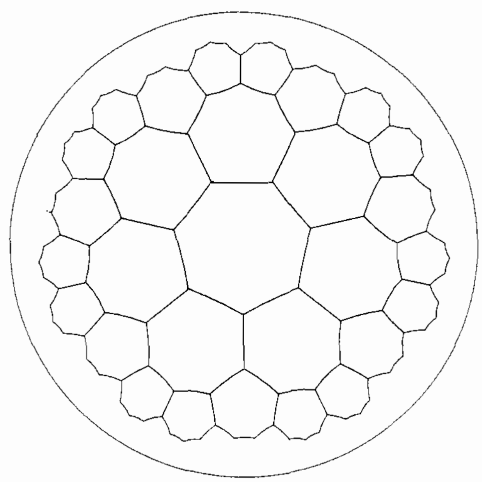


Figure 11.

The 0-th, first, and second layers of septagons of the tessellation {7,3}.

The pattern is extended from the k-th layer to the (k+1)-st layer by reflecting or rotating (depending on the group) the p-gon pattern across those edges and vertices common to both layers (Figure 12).

In theory, this stage of the replication process could be continued indefinitely, so that the hyperbolic plane would be filled with a repeating pattern. In practice, five layers are usually sufficient to give the appearance of filling a bounding circle less than a meter in diameter.

This process will be described in some detail for the groups [p,q] and [p,q]+ when p > 3 and q > 3. The other cases use similar, but slightly more complicated algorithms.

First, draw the p-gon pattern within the central p-gon. Assuming the existence of a procedure DrawPgonPattern which draws a transformed p-gon pattern given a transformation, this can be done by:

```
DrawPgonPattern(Identity)
```

Then if nLayers represents the number of layers to be drawn, the following algorithm will draw the remaining layers. RotateCenter, RotateVertx, and T are variable transformations (representing rotation about the centers and vertices of p-gons and a cumulative transformation, respectively); RotateP, RotateQ, and RotateEdge are constant rotations by angles of $2\pi/p$, $2\pi/q$, and π about the centers, vertices, and centers of the edges of the p-gons respectively. Multiplication of transformations is performed left to right.

```
RotateCenter := Identity;
FOR i := 1 TO p DO BEGIN
  T := RotateEdge * RotateCenter;
  ReplicatePattern(T, nLayers - 1, Edge);
  RotateCenter := RotateP * RotateCenter;
  RotateVertex := RotateQ * RotateP;
  FOR j := 1 TO q - 3 DO BEGIN
    ReplicatePattern(RotateVertex * T,
                     nLayers - 1, Vertex);
    RotateVertex := RotateQ * RotateVertex
  END
END
```

If nLayers is 1, ReplicatePattern merely calls DrawPgonPattern once--in this case, the above algorithm would draw the 0-th and first layers of p-gons. Figure 12 shows such a pattern. Figure 13 shows the patern extended to the second layer. Noting the similarity of the above algorithm to ReplicatePattern (below), it is easy to see that ReplicatePattern could be modified so that a single call to it would generate the entire pattern.

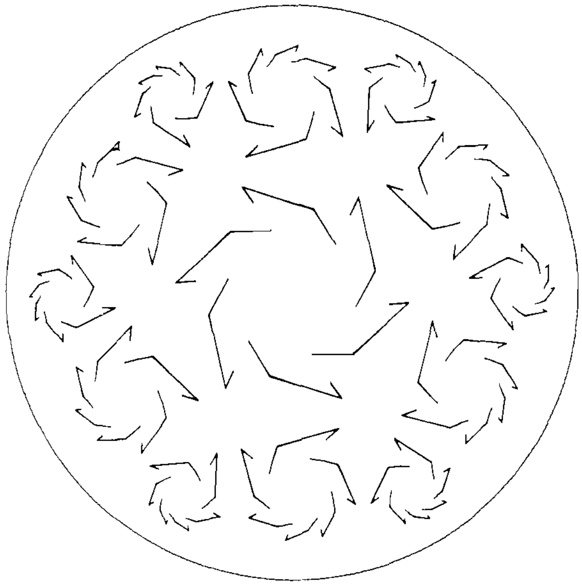


Figure 12.

This figure shows the p-gon pattern of Figure 10b replicated to fill the first layer of hexagons according to the symmetry group [6+,4].

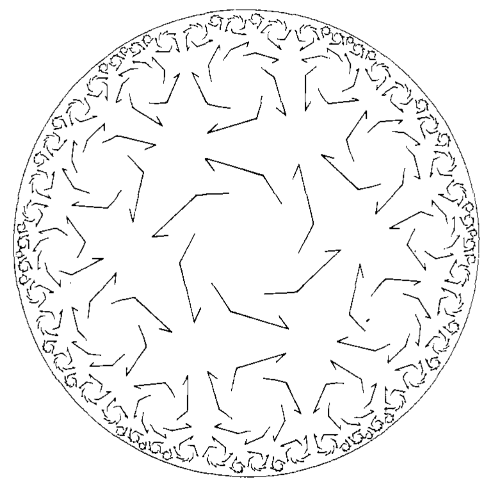


Figure 13.

The pattern of Figure 12 extended one more layer.

```
PROCEDURE ReplicatePattern
         (InitialTransform: trnasformation;
                         Layerdiff: integer;
                  Adjacency: connectivity);
VAR
  RotateCenter, RotateVertex, T:
                            transformation;
  i, ExposedEdges, j, PgonsPerVertex:
                                   integer;
BEGIN
  DrawPgonPattern(InitialTransform);
  IF Layerdiff > 0 THEN BEGIN
    CASE Adjacency OF
      Edge: ExposedEdges := p - 3;
      Vertex: ExposedEdges := p - 2
    END;
    RotateCenter := RotateP *
                    RotateP *
                    InitialTransform;
    FOR i := 1 TO ExposedEdges DO BEGIN
      T := RotateEdge * RotateCenter;
      ReplicatePattern(T, Layerdiff - 1,
                                    Edge);
      RotateCenter := RotateP *
                      RotateCenter;
      IF i < ExposedEdges THEN
        PgonsPerVertex := q - 3
      ELSE IF i = ExposedEdges THEN
        PgonsPerVertex := q - 4;
      RotateVertex := RotateQ * RotateP;
      FOR j := 1 TO PgonsPerVertex DO BEGIN
        ReplicatePattern(RotateVertex * T,
             Layerdiff - 1, Vertex);
        RotateVertex := RotateQ *
                        RotateVertex
      END
    END
  END;
```

See the Appendix for a description of DrawPgonPattern and the transformations RotateP, RotateQ, and RotateEdge.

6. RESULTS

This program can produce repeating hyperbolic patterns in color with any of the four symmetry groups described in Section 2. The motif boundary procedure allows for the creation of completely interlocking patterns. Figures 3 and 14, with symmetry groups [6,4] and [4,5] respectively, are samples of patterns with symmetry groups of the form [p,q]. The pattern of Figure 15 has symmetry group [5,4]+. To our knowledge, this is the first pattern more complicated than the half-shaded, half-blank patterns mentioned in Section 3 (e. g. Figure 6) which has been created with or without computer aid and which has symmetry group of the form [p,q]+. Figure 13 shows a pattern with symmetry group [6+,4]. Again, to our knowledge, no other pattern has been created having a symmetry group of the form [p+,q]. Figure 16 shows a duplication (ignoring shading) of M. C. Escher's "Circle Limit II" (Figure 7) and Figure 17 shows and outline of his "Circle Limit IV" (Figure 8), both having symmetry groups of the form [p,q+].

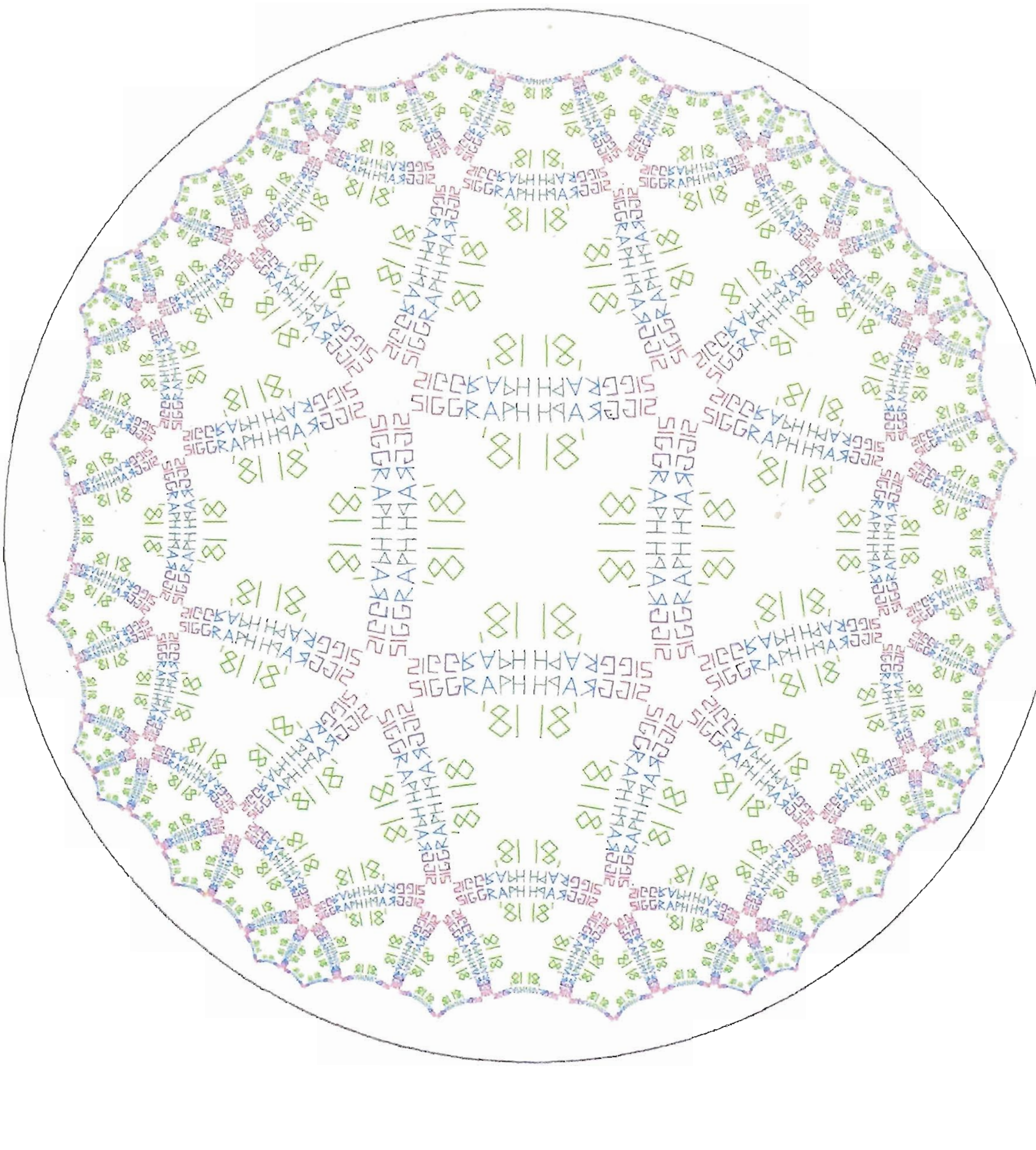


Figure 14. A pattern with symmetry group [4,5].

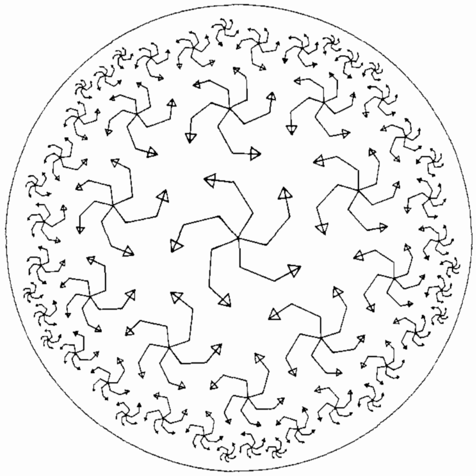


Figure 15. A pattern with symmetry group [5,4]+.

7. EXTENSIONS

First, the program could be extended to include more complicated symmetry groups, such as the symmetry groups of M. C. Escher's other hyperbolic patterns, "Circle Limit I" and "Circle Limit III". Next, the program could allow for color symmetry, i. e. the colors would be permuted by successive transformations of the motif. Escher's "Circle Limit II" and "Circle Limit III" are examples of such color symmetry.

Another natural extension would be to allow for the construction of motifs out of shaded polygons--the final pattern being dispalyed on an area-oriented output device such as a raster CRT. The motif boundary procedure could easily be modified to handle polygons.

Many of the techniques used in creating hyperbolic patterns could also be used to create repeating patterns of the Euclidean plane or the sphere.

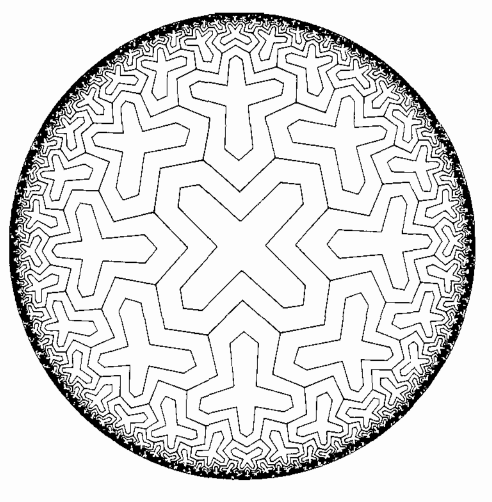


Figure 16.

A duplication of the pattern of M. C. Escher's Circle Limit II (Figure 7), having symmetry group [8,3+].

8. SUMMARY

This program can create pleasing repeating patterns of the hyperbolic plane in a few minutes, a process that would require months to complete to the same precision without computer aid. Only the Dutch artist M. C.Escher had the patience to create such patterns by hand. The process of creating these patterns brings together the disciplines of computer science, art, and mathematics. This is a useful educational tool for the illustration of concepts in transformation group theory and hyperbolic geometry.

The computer program which generated these figures is written in FORTRAN, using Tektronics and Zeta supporting software.

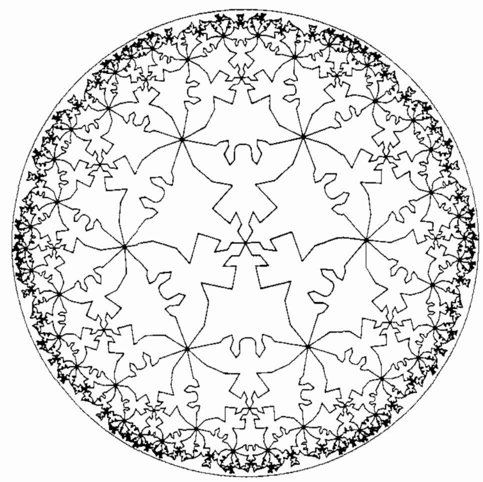


Figure 17.

A duplication of the pattern of M. C. Escher's Circle Limit IV (Figure 8), having symmetry group [6,4+].

APPENDIX

The transformations used in our program are represented by 3-by-3 real matrices. For instance, reflections across the sides of the triangular fundamental region for the group [p,q] can be represented by:

$$\text{ReflectEdgeBisector} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{ReflectPgonEdge} := \begin{bmatrix} -\cosh(2b) & 0 & \sinh(2b) \\ 0 & 1 & 0 \\ -\sinh(2b) & 0 & \cosh(2b) \end{bmatrix}$$

$$\text{ReflectHypotenuse} := \begin{bmatrix} \cos(2\ /p) & \sin(2\ /p) & 0 \\ \sin(2\ /p) & -\cos(2\ /p) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $\cosh(b) = \cos(\pi/q)\ /\ \sin(\pi/p)$

$\cosh(2b) = 2 * \cosh(b)**2 - 1$

$\sinh(2b) = \text{sqrt}(\cosh(2b)**2 - 1)$

The transformations RotateP, RotateQ, and RotateEdge are given by:

```
RotateP := ReflectEdgeBisector *
               ReflectHypotenuse
RotateQ := ReflectHypotenuse *
               ReflectPgonEdge
RotateEdge := ReflectPgonEdge *
               ReflectEdgeBisector
```

The DrawPgonPattern algorithm can be described as follows:

```
PROCEDURE DrawPgonPattern(T: transformation);

  VAR
    SumSquare, Tx, Ty: real;
    Z: ARRAY[1..3] OF real;

  BEGIN
    FOR i := 1 TO nPgonActions DO BEGIN
      SumSquare := X[i]*X[i] + Y[i]*Y[i];
      Z[1] := 2 * X[i] / (1 - SumSquare);
      Z[2] := 2 * Y[i] / (1 - SumSquare);
      Z[3] := (1 + SumSquare)/(1 - SumSquare);
      Z := T * Z;
      Tx := Z[1]/(1 + Z[3]);
      Ty := Z[2]/(1 + Z[3]);
      CASE Action[i] OF
        Move: MoveTo(Tx, Ty);
        Draw: LineTo(Tx, Ty);
        Black: Color := 'Black';
        Blue: Color := 'Blue';
        Red: Color := 'Red';
        Yellow: Color := 'Yellow'
      END
    END
  END;
```

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