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DIVERGENT TORUS ORBITS IN HOMOGENEOUS SPACES OF $\mathbb{Q}$-RANK TWO

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ABSTRACT

Let $\mathbf{G}$ be a semisimple algebraic $\mathbb{Q}$-group, let Γ be an arithmetic subgroup of $\mathbf{G}$, and let $\mathbf{T}$ be an $\mathbb{R}$-split torus in $\mathbf{G}$. We prove that if there is a divergent $\mathbf{T}_{\mathbb{R}}$-orbit in $\Gamma\backslash\mathbf{G}_{\mathbb{R}}$, and $\mathbb{Q}\text{-rank}\,\mathbf{G} \leq 2$, then $\dim \mathbf{T} \leq \mathbb{Q}\text{-rank}\,\mathbf{G}$. This provides a partial answer to a question of G. Tomanov and B. Weiss.

1. Introduction

Let $\mathbf{G}$ be a semisimple algebraic $\mathbb{Q}$-group, let Γ be an arithmetic subgroup of $\mathbf{G}$, and let $\mathbf{T}$ be an $\mathbb{R}$-split torus in $\mathbf{G}$. The $\mathbf{T}_{\mathbb{R}}$-orbit of a point Γx_0 in $X = \Gamma\backslash\mathbf{G}_{\mathbb{R}}$ is **divergent** if the natural orbit map $\mathbf{T}_{\mathbb{R}} \to X\colon t \mapsto \Gamma x_0 t$ is proper. G. Tomanov and B. Weiss [TW, p. 389] asked whether it is possible

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for there to be a divergent $\mathbf{T}_{\mathbb{R}}$-orbit when $\dim \mathbf{T} > \mathbb{Q}\text{-rank}\,\mathbf{G}$. B. Weiss [W1, Conjecture 4.11A] conjectured that the answer is negative.

1.1. CONJECTURE: *Let*

- $\mathbf{G}$ *be a semisimple algebraic group that is defined over* $\mathbb{Q}$,
- Γ *be a subgroup of* $\mathbf{G}_{\mathbb{R}}$ *that is commensurable with* $\mathbf{G}_{\mathbb{Z}}$,
- T *be a connected Lie subgroup of an* $\mathbb{R}$*-split torus in* $\mathbf{G}_{\mathbb{R}}$, *and*
- $x_0 \in \mathbf{G}_{\mathbb{R}}$.

If the T*-orbit of* Γx_0 *is divergent in* $\Gamma\backslash\mathbf{G}_{\mathbb{R}}$, *then* $\dim T \leq \mathbb{Q}\text{-rank}\,\mathbf{G}$.

The conjecture easily reduces to the case where $\mathbf{G}$ is connected and $\mathbb{Q}$-simple. Furthermore, the desired conclusion is obvious if $\mathbb{Q}\text{-rank}\,\mathbf{G} = 0$ (because this implies that $\Gamma\backslash\mathbf{G}_{\mathbb{R}}$ is compact), and it is easy to prove if $\mathbb{Q}\text{-rank}\,\mathbf{G} = 1$ (see §2). Our main result is that the conjecture is also true in the first interesting case:

1.2. THEOREM: *Suppose* $\mathbf{G}$, Γ, T, *and* x_0 *are as specified in Conjecture 1.1, and assume* $\mathbb{Q}\text{-rank}\,\mathbf{G} \leq 2$. *If the* T*-orbit of* Γx_0 *is divergent in* $\Gamma\backslash\mathbf{G}_{\mathbb{R}}$, *then* $\dim T \leq \mathbb{Q}\text{-rank}\,\mathbf{G}$.

The proof is based on the fact that if f is any continuous map from the 2-sphere S^2 to any simplicial complex Σ^k of dimension $k < 2$, then there exist two antipodal points x and y of S^2, such that $f(x) = f(y)$.

For higher $\mathbb{Q}$-ranks, we prove only the upper bound $\dim T < 2(\mathbb{Q}\text{-rank}\,\mathbf{G})$ (see 6.1). The factor of 2 in this bound is due to the existence of maps $f\colon S^n \to \Sigma^k$, with $k = \lceil (n+1)/2 \rceil$, such that no two antipodal points of S^n have the same image in Σ^k (see 6.3).

The first partial result on the conjecture was proved by G. Tomanov and B. Weiss [TW, Theorem 1.4], who showed that if $\mathbb{Q}\text{-rank}\,\mathbf{G} < \mathbb{R}\text{-rank}\,\mathbf{G}$, then $\dim T < \mathbb{R}\text{-rank}\,\mathbf{G}$. After seeing a preliminary version of our work, B. Weiss [W2] has recently proved the conjecture in all cases.

GEOMETRIC REFORMULATION. We remark that, by using the well-known fact that flats in a symmetric space of noncompact type are orbits of $\mathbb{R}$-split tori in its isometry group [H, Proposition 6.1, p. 209], the conjecture and our theorem can also be stated in the following geometric terms.

Suppose $\widetilde{X}$ is a symmetric space, with no Euclidean (local) factors. Recall that a **flat** in $\widetilde{X}$ is a connected, totally geodesic, flat submanifold of $\widetilde{X}$. Up to isometry, $\widetilde{X} = G/K$, where K is a compact subgroup of a connected, semisimple Lie group G with finite center. Then $\mathbb{R}\text{-rank}\,G$ has the following geometric interpretation:

1.3. FACT: $\mathbb{R}$-rank G is the largest natural number r, such that $\widetilde{X}$ contains a topologically closed, simply connected, r-dimensional flat.

Now let $X = \Gamma\backslash\widetilde{X}$ be a locally symmetric space modeled on X, and assume that X has finite volume. Then $\mathbb{Q}$-rank Γ is a certain algebraically defined invariant of Γ [M, §9D]. It can be characterized by the following geometric property:

1.4. PROPOSITION: *$\mathbb{Q}$-rank Γ is the smallest natural number r, for which there exists collection of finitely many r-dimensional flats in X, such that all of X is within a bounded distance of the union of these flats.*

It is clear from this that the $\mathbb{Q}$-rank does not change if X is replaced by a finite cover, and that it satisfies $\mathbb{Q}$-rank $\Gamma \leq \mathbb{R}$-rank G. Furthermore, the algebraic definition easily implies that if $\mathbb{Q}$-rank $\Gamma = r$, then some finite cover of X contains a topologically closed, simply connected flat of dimension r. If Conjecture 1.1 is true, then there are no such flats of larger dimension. In other words, $\mathbb{Q}$-rank should have the following geometric interpretation, analogous to (1.3):

1.5. CONJECTURE: *$\mathbb{Q}$-rank Γ is the largest natural number r, such that some finite cover of X contains a topologically closed, simply connected, r-dimensional flat.*

More precisely, Conjecture 1.1 is equivalent to the assertion that $\mathbb{Q}$-rank Γ is the largest natural number r, such that $\widetilde{X}$ contains a topologically closed, simply connected, r-dimensional flat F, for which the composition $F \hookrightarrow \widetilde{X} \to X$ is a proper map.

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2. Example: A proof for $\mathbb{Q}$-rank 1

To illustrate the ideas in our proof of Theorem 1.2, we sketch a simple proof that applies when $\mathbb{Q}$-rank $\mathbf{G} = 1$. (A similar proof appears in [W1, Proposition 4.12].)

Proof: Suppose $\mathbf{G}$, Γ, T, and x_0 are as specified in Conjecture 1.1. For convenience, let $\pi\colon \mathbf{G}_{\mathbb{R}} \to \Gamma\backslash\mathbf{G}_{\mathbb{R}}$ be the natural covering map. Assume that $\mathbb{Q}$-rank $\mathbf{G} = 1$, that $\dim T = 2$, and that the T-orbit of $\pi(x_0)$ is divergent in $\Gamma\backslash G$. This will lead to a contradiction.

Let $E_1 = \Gamma\backslash \mathbf{G}_{\mathbb{R}}$. Because $\mathbb{Q}$-rank $\mathbf{G} = 1$, reduction theory (the theory of Siegel sets) implies that there exist

- a compact subset E_0 of $\Gamma\backslash \mathbf{G}_{\mathbb{R}}$, and
- a $\mathbb{Q}$-representation $\rho\colon \mathbf{G} \to \mathbf{GL}_m$ (for some m),

such that, for each connected component $\mathcal{E}$ of $G_{\mathbb{R}} \smallsetminus \pi^{-1}(E_0)$, there is a nonzero vector $v \in \mathbb{Q}^m$, such that

$$\text{(2.1)} \qquad \text{if } \lim_{n\to\infty} \Gamma g_n = \infty \text{ in } \Gamma\backslash \mathbf{G}_{\mathbb{R}}, \text{ and } \{g_n\} \subset \mathcal{E}, \text{ then } \lim_{n\to\infty} \rho(g_n)v = 0.$$

(In geometric terms, this is the fact that, because $E_1 \smallsetminus E_0$ consists of disjoint "cusps," $\mathbf{G}_{\mathbb{R}} \smallsetminus \pi^{-1}(E_0)$ consists of disjoint "horoballs.")

Given $\epsilon > 0$, let T_R be a large circle (1-sphere) in T, centered at the identity element. Because the T-orbit of $\pi(x_0)$ is divergent, we may assume $\pi(x_0 T_R)$ is disjoint from E_0. Then, because $T_R \approx S^1$ is connected, the set $x_0 T_R$ must be contained in a single component of $G_{\mathbb{R}} \smallsetminus \pi^{-1}(E_0)$. Thus, there is a vector $v \in \mathbb{Q}^n$, such that $\|\rho(t)v\| < \epsilon\|v\|$ for all $t \in T_R$.

Fix $t \in T_R$. Then t^{-1} also belongs to T_R, so $\|\rho(t)v\|$ and $\|\rho(t^{-1})v\|$ are both much smaller than $\|v\|$. This is impossible (see 3.2). ∎

The above proof does not apply directly when $\mathbb{Q}$-rank $\mathbf{G} = 2$, because, in this case, there are arbitrarily large compact subsets C of $\Gamma\backslash \mathbf{G}_{\mathbb{R}}$, such that $\mathbf{G}_{\mathbb{R}} \smallsetminus \pi^{-1}(C)$ is connected. Instead of only E_0 and E_1, we consider a more refined stratification $E_0 \subset E_1 \subset E_2$ of $\Gamma\backslash G$. (It is provided by the structure of Siegel sets in $\mathbb{Q}$-rank two.) The set E_0 is compact, and, for $i \geq 1$, each component $\mathcal{E}$ of $\pi^{-1}(E_i \smallsetminus E_{i-1})$ has a corresponding representation ρ and vector v, such that (2.1) holds. Thus, it suffices to find a component of either $\pi^{-1}(E_1 \smallsetminus E_0)$ or $\pi^{-1}(E_2 \smallsetminus E_1)$ that contains two antipodal points of T_R. Actually, we replace E_1 with a slightly larger set that is open, so that we may apply the following property of S^2:

2.2. Proposition (see 3.1): *Suppose $n \geq 2$, and that $\{V_1, V_2\}$ is an open cover of the n-sphere S^n that consists of only 2 sets. Then there is a connected component C of some V_i, such that C contains two antipodal points of S^n.*

2.3. *Remark:* In §5, we do not use the notation $E_0 \subset E_1 \subset E_2$. The role of E_0 is played by $\pi(Q\mathcal{S}_{\Delta}^{+})$, the role of an open set containing E_1 is played by $\pi(Q\mathcal{S}_{\alpha} \cup Q\mathcal{S}_{\beta})$, and the role of $E_2 \smallsetminus E_1$ is played by $\pi(Q\mathcal{S}_{*})$.

3. Preliminaries

The classical Borsuk–Ulam Theorem implies that if $f\colon S^n \to \mathbb{R}^k$ is a continuous map, and $n \geq k$, then there exist two antipodal points x and y of S^n, such that $f(x) = f(y)$. We use this to prove the following stronger version of Proposition 2.2:

3.1. Proposition: *Suppose $\mathcal{V}$ is an open cover of S^n, with $n \geq 2$, such that no point of S^n is contained in more than two of the sets in $\mathcal{V}$. Then some $V \in \mathcal{V}$ contains two antipodal points of S^n.*

Proof: Because S^n is compact, we may assume the open cover $\mathcal{V}$ is finite. Let $\{\phi_V\}_{V\in\mathcal{V}}$ be a partition of unity subordinate to $\mathcal{V}$. This naturally defines a continuous function Φ from S^n to the simplex

$$\Delta_{\mathcal{V}} = \left\{ (x_V)_{V\in\mathcal{V}} \,\middle|\, \sum_{V\in\mathcal{V}} x_V = 1 \right\} \subset [0,1]^{\mathcal{V}}.$$

Namely, $\Phi(x) = (\phi_V(x))_{V\in\mathcal{V}}$. Our hypothesis on $\mathcal{V}$ implies that no more than two components of $\Phi(x)$ are nonzero, so the image of Φ is contained in the 1-skeleton $\Delta_{\mathcal{V}}^{(1)}$ of $\Delta_{\mathcal{V}}$. Because S^n is simply connected, Φ lifts to a map $\widetilde{\Phi}$ from S^n to the universal cover $\widetilde{\Delta_{\mathcal{V}}^{(1)}}$ of $\Delta_{\mathcal{V}}^{(1)}$. The universal cover is a tree, which can be embedded in $\mathbb{R}^2$, so the Borsuk–Ulam Theorem implies that there exist two antipodal points x and y of S^n, such that $\widetilde{\Phi}(x) = \widetilde{\Phi}(y)$. Thus, there exists $V \in \mathcal{V}$, such that $\phi_V(x) = \phi_V(y) \neq 0$. So $x, y \in V$. ∎

For completeness, we also provide a proof of the following simple observation.

3.2. Lemma: *Let T be any abelian group of diagonalizable $n \times n$ real matrices. There is a constant $\epsilon > 0$, such that if*

- *v is any vector in $\mathbb{R}^n$, and*
- *t is any element of T,*

then either $||tv|| \geq \epsilon ||v||$ or $||t^{-1}v|| \geq \epsilon ||v||$.

Proof: The elements of T can be simultaneously diagonalized. Thus, after a change of basis (which affects norms by only a bounded factor), we may assume that each standard basis vector e_i is an eigenvector for every element of T.

Write $v = (v_1, \ldots, v_n)$, and let t_i be the eigenvalue of t corresponding to the eigenvector e_i. Because any two norms differ only by a bounded factor, we may assume $||\ ||$ is the sup norm on $\mathbb{R}^n$; therefore, we have $||v|| = |v_j|$ for some j. We may assume $|t_j| \geq 1$, by replacing t with t^{-1} if necessary. Then

$$||tv|| = ||(t_1 v_1, \ldots, t_n v_n)|| \geq |t_j v_j| = |t_j| \cdot ||v|| \geq ||v||,$$

as desired. ∎

4. Properties of Siegel sets

We present some basic results from reduction theory that follow easily from the fundamental work of A. Borel and Harish-Chandra [BH] (see also [B, §13–§15]). Most of what we need is essentially contained in [L, §2], but we are working in G, rather than in $\widetilde{X} = G/K$. We begin by setting up the standard notation.

4.1. Notation (cf. [L, §1])*:* Let

- $\mathbf{G}$ be a connected, almost simple $\mathbb{Q}$-group, with $\mathbb{Q}\text{-rank}\,\mathbf{G} = 2$,
- G be the identity component of $\mathbf{G}_\mathbb{R}$,
- Γ be a finite-index subgroup of $G_\mathbb{Z} \cap G$,
- P be a minimal parabolic $\mathbb{Q}$-subgroup of G,
- $\mathbf{A}$ be a maximal $\mathbb{Q}$-split torus of $\mathbf{G}$,
- A be the identity component of $\mathbf{A}_\mathbb{R}$, and
- K be a maximal compact subgroup of G.

We may assume $A \subset P$. Then we have a Langlands decomposition $P = UMA$, where U is unipotent and M is reductive. We remark that U and A are connected, but M is not connected (because P is not connected).

4.2. Notation (cf. [L, §1])*:* The choice of P determines an ordering of the $\mathbb{Q}$-roots of $\mathbf{G}$. Because $\mathbb{Q}\text{-rank}\, G = 2$, there are precisely two simple $\mathbb{Q}$-roots α and β (so the base Δ is $\{\alpha, \beta\}$). Then α and β are homomorphisms from A to $\mathbb{R}^+$.

Any element g of G can be written in the form $g = pak$, with $p \in UM$, $a \in A$, and $k \in K$. The element a is uniquely determined by g, so we may use this decomposition to extend α and β to continuous functions $\tilde{\alpha}$ and $\tilde{\beta}$ defined on all of G:

$$\tilde{\alpha}(g) = \alpha(a) \quad \text{if } g \in UMaK \text{ and } a \in A,$$
$$\tilde{\beta}(g) = \beta(a) \quad \text{if } g \in UMaK \text{ and } a \in A.$$

4.3. Notation (cf. [L, §2])*:*

- Fix a subset Q of $\mathbf{G}_\mathbb{Q} \cap G$, such that

$$Q \text{ is a set of representatives of } \Gamma \backslash (\mathbf{G}_\mathbb{Q} \cap G)/(\mathbf{P}_\mathbb{Q} \cap P).$$

  Note that Q is finite.
- For $\tau > 0$, let $A_\tau = \{a \in A \mid \alpha(a) > \tau \text{ and } \beta(a) > \tau\}$.

- For $\tau > 0$ and a precompact, open subset ω of UM, let $\mathcal{S}_{\tau,\omega} = \omega A_\tau K$. This is a **Siegel set** in G.
- We fix $\tau > 0$ and a precompact, open subset ω of UM, such that, letting $\mathcal{S} = \mathcal{S}_{\tau,\omega}$, we have

$$Q\mathcal{S} \text{ is a fundamental set for } \Gamma \text{ in } G.$$

  That is,
  - $\Gamma Q\mathcal{S} = G$, and
  - $\{\gamma \in \Gamma \mid \gamma Q\mathcal{S} \cap pQ\mathcal{S} \neq \emptyset\}$ is finite, for all $p \in G_{\mathbb{Q}} \cap G$.
- Let $\mathcal{D} = \left\{ p^{-1}\gamma q \,\middle|\, \begin{array}{c} p,q \in Q, \gamma \in \Gamma, \\ p\mathcal{S} \cap \gamma q\mathcal{S} \text{ is } \textit{not} \text{ precompact} \end{array} \right\} \subset \mathbf{G}_{\mathbb{Q}} \cap G$. Note that $\mathcal{D}$ is finite.
- Fix $r > 0$, such that, for $q \in \mathcal{D}$, we have
  - if $\tilde{\alpha}$ is bounded on $\mathcal{S} \cap q\mathcal{S}$, then $\tilde{\alpha}(\mathcal{S} \cap q\mathcal{S}) < r$, and
  - if $\tilde{\beta}$ is bounded on $\mathcal{S} \cap q\mathcal{S}$, then $\tilde{\beta}(\mathcal{S} \cap q\mathcal{S}) < r$.
- Fix any $r^* > r$.
- Define
  - $\mathcal{S}_* = \{x \in \mathcal{S} \mid \tilde{\alpha}(x) > r \text{ and } \tilde{\beta}(x) > r\}$,
  - $\mathcal{S}_\alpha = \{x \in \mathcal{S} \mid \tilde{\alpha}(x) < r^*\}$,
  - $\mathcal{S}_\beta = \{x \in \mathcal{S} \mid \tilde{\beta}(x) < r^*\}$, and
  - $\mathcal{S}_\Delta = \mathcal{S}_\alpha \cap \mathcal{S}_\beta$.

  Note that $\{\mathcal{S}_*, \mathcal{S}_\alpha, \mathcal{S}_\beta\}$ is an open cover of $\mathcal{S}$ (whereas [L, p. 398] defines $\{\mathcal{S}_*, \mathcal{S}_\alpha, \mathcal{S}_\beta\}$ to be a partition of $\mathcal{S}$, so not all sets are open). We have

$$G = \Gamma Q\mathcal{S}_* \cup \Gamma Q\mathcal{S}_\alpha \cup \Gamma Q\mathcal{S}_\beta.$$

- For $p, q \in Q$, let

$$\begin{aligned} \mathcal{D}_0^{p,q} &= \{\gamma \in \Gamma \mid p\mathcal{S} \cap \gamma q\mathcal{S} \text{ is precompact and nonempty}\}, \\ \mathcal{D}_\alpha^{p,q} &= \{\gamma \in \Gamma \mid p\mathcal{S}_\alpha \cap \gamma q\mathcal{S}_\alpha \text{ is precompact and nonempty}\}, \\ \mathcal{D}_\beta^{p,q} &= \{\gamma \in \Gamma \mid p\mathcal{S}_\beta \cap \gamma q\mathcal{S}_\beta \text{ is precompact and nonempty}\}, \\ \mathcal{D}_{\alpha,\beta}^{p,q} &= \{\gamma \in \Gamma \mid p\mathcal{S}_\alpha \cap \gamma q\mathcal{S}_\beta \text{ is precompact and nonempty}\}, \end{aligned}$$

  and, using an overline to denote the closure of a set,

$$\begin{aligned} \mathcal{S}_\Delta^+ = \bigcup_{\gamma \in \mathcal{D}_0^{p,q}} \left(\overline{p\mathcal{S} \cap \gamma q\mathcal{S}}\right) \cup \bigcup_{\gamma \in \mathcal{D}_\alpha^{p,q}} \left(\overline{p\mathcal{S}_\alpha \cap \gamma q\mathcal{S}_\alpha}\right) \\ \cup \bigcup_{\gamma \in \mathcal{D}_\beta^{p,q}} \left(\overline{p\mathcal{S}_\beta \cap \gamma q\mathcal{S}_\beta}\right) \cup \bigcup_{\gamma \in \mathcal{D}_{\alpha,\beta}^{p,q}} \left(\overline{p\mathcal{S}_\alpha \cap \gamma q\mathcal{S}_\beta}\right). \end{aligned}$$

Note that $\mathcal{D}_0^{p,q}$, $\mathcal{D}_\alpha^{p,q}$, $\mathcal{D}_\beta^{p,q}$, and $\mathcal{D}_0^{p,q}$ are finite (because $Q\mathcal{S}$ is a fundamental set), so $\mathcal{S}_\Delta^+$ is compact. And $\Gamma\mathcal{S}_\Delta^+$ is closed.

- For $\Theta \subset \Delta$, we use P_Θ to denote the corresponding standard parabolic $\mathbb{Q}$-subgroup of G corresponding to Θ. In particular, $P_\emptyset = P$ and $P_\Delta = G$. There is a corresponding Langlands decomposition $P_\Theta = U_\Theta M_\Theta A_\Theta$.

We now state two propositions from [L], that we will use repeatedly in the proofs of the next few lemmas. These propositions hold more generally for semisimple $\mathbb{Q}$-algebraic groups of arbitrary $\mathbb{Q}$-rank.

4.4. Proposition ([L, Proposition 2.3]): *Let $p, q \in Q$ and $\gamma \in \Gamma$, such that the intersection $p\mathcal{S} \cap \gamma q\mathcal{S}$ is not precompact. Then $p^{-1}\gamma q \in P_\Theta \cap G_\mathbb{Q}$ where Θ is the collection of all the roots $\lambda \in \Delta$ for which $\tilde{\lambda}(\mathcal{S} \cap p^{-1}\gamma q\mathcal{S})$ is bounded.*

4.5. Proposition ([L, Lemma 2.4(i)]): *For all $\gamma, \tilde{\gamma} \in \Gamma$ and $p, q \in Q$, we have:*
(1) *If $p^{-1}\gamma q \in P$, then $p = q$ and $p^{-1}\gamma q \in (UM)_\mathbb{Q}$.*
(2) *Let $\Theta \subset \Delta$. If both $p^{-1}\gamma q$ and $p^{-1}\tilde{\gamma} q$ are in P_Θ, then*

$$(p^{-1}\gamma q)^{-1} p^{-1}\tilde{\gamma} q = q^{-1}\gamma^{-1}\tilde{\gamma} q \in (U_\Theta M_\Theta)_\mathbb{Q}.$$

4.6. Lemma: *For all $\gamma \in \Gamma$ and $p, q \in Q$, we have:*
(1) *$p\mathcal{S}_\alpha \cap \gamma q\mathcal{S}_\beta$ is precompact, and*
(2) *$p\mathcal{S}_\alpha \cap \gamma q\mathcal{S}_\beta \subset \mathcal{S}_\Delta^+$.*

Proof: It suffices to prove (1), for then (2) is immediate from the definition of $\mathcal{S}_\Delta^+$ (and $\mathcal{D}_{\alpha,\beta}^{p,q}$). Thus, let us suppose that $p\mathcal{S}_\alpha \cap \gamma q\mathcal{S}_\beta$ is not precompact. This will lead to a contradiction.

Because $\tilde{\alpha}$ is bounded on $\mathcal{S}_\alpha$, but $\mathcal{S}_\alpha \cap p^{-1}\gamma q\mathcal{S}_\beta$ is not precompact, we know that $\tilde{\beta}$ is unbounded on $\mathcal{S}_\alpha \cap p^{-1}\gamma q\mathcal{S}_\beta$ (and, hence, on $\mathcal{S} \cap p^{-1}\gamma q\mathcal{S}$). Therefore, Proposition 4.4 implies that

$$p^{-1}\gamma q \in P_\alpha.$$

Similarly (replacing γ with γ^{-1} and interchanging p with q and α with β), because $\gamma^{-1}p\mathcal{S}_\alpha \cap q\mathcal{S}_\beta = \gamma^{-1}(p\mathcal{S}_\alpha \cap \gamma q\mathcal{S}_\beta)$ is not precompact, we see that

$$q^{-1}\gamma^{-1}p \in P_\beta.$$

Noting that $q^{-1}\gamma^{-1}p = (p^{-1}\gamma q)^{-1}$, we conclude that $p^{-1}\gamma q \in P_\alpha \cap P_\beta = P_\emptyset$, so Proposition 4.5 (1) tells us that $p = q$ and $p^{-1}\gamma q \in UM$. Therefore

$$\tilde{\alpha}(\mathcal{S}_\alpha \cap p^{-1}\gamma q\mathcal{S}_\beta) \subset \tilde{\alpha}(\mathcal{S}_\alpha)$$

and

$$\tilde{\beta}(\mathcal{S}_\alpha \cap p^{-1}\gamma q \mathcal{S}_\beta) \subset \tilde{\beta}(p^{-1}\gamma q\mathcal{S}_\beta) \subset \tilde{\beta}(UM\mathcal{S}_\beta) = \tilde{\beta}(\mathcal{S}_\beta)$$

are precompact. So $\mathcal{S}_\alpha \cap p^{-1}\gamma q\mathcal{S}_\beta$ is precompact, which contradicts our assumption that $p\mathcal{S}_\alpha \cap \gamma q\mathcal{S}_\beta$ is not precompact. ∎

4.7. Lemma: *If $\gamma \in \Gamma$ and $p, q \in Q$, such that $p\mathcal{S}_* \cap \gamma q\mathcal{S}_* \not\subset \mathcal{S}^+_\Delta$, then $p = q$ and $p^{-1}\gamma q \in (UM)_\mathbb{Q}$.*

Proof: It suffices to show that both $\tilde{\alpha}$ and $\tilde{\beta}$ are unbounded on $\mathcal{S} \cap p^{-1}\gamma q\mathcal{S}$, for then the desired conclusion is obtained from Proposition 4.4 and Proposition 4.5 (1). Thus, let us suppose (without loss of generality) that

$$\tilde{\alpha} \text{ is bounded on } \mathcal{S} \cap p^{-1}\gamma q\mathcal{S}.$$

This will lead to a contradiction.

Case 1: Assume $\tilde{\beta}$ is also bounded on $\mathcal{S} \cap p^{-1}\gamma q\mathcal{S}$. Then $p\mathcal{S} \cap \gamma q\mathcal{S} = p(\mathcal{S} \cap p^{-1}\gamma q\mathcal{S})$ is precompact, so, by definition, $p\mathcal{S} \cap \gamma q\mathcal{S} \subset \mathcal{S}^+_\Delta$. Therefore

$$p\mathcal{S}_* \cap \gamma q\mathcal{S}_* \subset p\mathcal{S} \cap \gamma q\mathcal{S} \subset \mathcal{S}^+_\Delta.$$

This contradicts the hypothesis of the lemma.

Case 2: Assume $\tilde{\beta}$ is not bounded on $\mathcal{S} \cap p^{-1}\gamma q\mathcal{S}$. As $\tilde{\alpha}$ is bounded on $\mathcal{S} \cap p^{-1}\gamma q\mathcal{S}$, from the definition of $\mathcal{S}_\alpha$, we see that $p\mathcal{S} \cap \gamma q\mathcal{S} \subset p\mathcal{S}_\alpha$. Therefore

$$p\mathcal{S}_* \cap \gamma q\mathcal{S}_* \subset p\mathcal{S}_* \cap p\mathcal{S}_\alpha = \emptyset \subset \mathcal{S}^+_\Delta.$$

This contradicts the hypothesis of the lemma. ∎

4.8. Corollary: *If x and y are two points in the same connected component of $\Gamma Q\mathcal{S}_* \smallsetminus \Gamma\mathcal{S}^+_\Delta$, then there exist $\gamma_0, \gamma \in \Gamma$ and $q \in Q$, such that $x \in \gamma_0 q\mathcal{S}_*$, $y \in \gamma_0\gamma q\mathcal{S}_*$, and $q^{-1}\gamma q \in (UM)_\mathbb{Q}$.*

4.9. Lemma:

(1) *If $\gamma \in \Gamma$ and $p, q \in Q$, such that $p\mathcal{S}_\alpha \cap \gamma q\mathcal{S}_\alpha \not\subset \mathcal{S}^+_\Delta$, then $p^{-1}\gamma q \in (P_\alpha)_\mathbb{Q}$.*

(2) *For each $p, q \in Q$, there exists $h_{p,q} \in (P_\alpha)_\mathbb{Q}$, such that $p^{-1}\Gamma q \cap (P_\alpha)_\mathbb{Q} \subset h_{p,q}(U_\alpha M_\alpha)_\mathbb{Q}$.*

Proof: (1) Because $p\mathcal{S}_\alpha \cap \gamma q\mathcal{S}_\alpha \not\subset \mathcal{S}^+_\Delta$, we know from the definition of $\mathcal{S}^+_\Delta$ (and $\mathcal{D}^{p,q}_\alpha$) that $p\mathcal{S}_\alpha \cap \gamma q\mathcal{S}_\alpha$ is not precompact. Since $\tilde{\alpha}$ is bounded on $\mathcal{S}_\alpha$, we conclude that $\tilde{\beta}$ is **not** bounded on $\mathcal{S}_\alpha \cap p^{-1}\gamma q \mathcal{S}_\alpha$ (and, hence, on $\mathcal{S} \cap p^{-1}\gamma q \mathcal{S}$). Then Proposition 4.4 asserts that $p^{-1}\gamma q \in (P_\Theta)_{\mathbb{Q}}$, for $\Theta = \{\alpha\}$ or $\emptyset$. Because $P_\emptyset \subset P_\alpha$, we conclude that $p^{-1}\gamma q \in (P_\alpha)_{\mathbb{Q}}$.

(2) From Proposition 4.5 (2), we see that the coset $(p^{-1}\gamma q)(U_\alpha M_\alpha)_{\mathbb{Q}}$ does not depend on the choice of γ, if we require γ to be an element of Γ, such that $p^{-1}\gamma q \in (P_\alpha)_{\mathbb{Q}}$. ∎

4.10. COROLLARY: *If x and y are two points in the same connected component of $\Gamma Q \mathcal{S}_\alpha \smallsetminus \Gamma \mathcal{S}^+_\Delta$, then there exist $\gamma_0, \gamma \in \Gamma$ and $p, q \in Q$, such that $x \in \gamma_0 p \mathcal{S}_\alpha$, $y \in \gamma_0 \gamma q \mathcal{S}_\alpha$, and $p^{-1}\gamma q \in h_{p,q}(U_\alpha M_\alpha)_{\mathbb{Q}}$.*

5. Proof of the Main Theorem

Let G, Γ, T and x_0 be as described in the hypotheses of Theorem 1.2, and assume $\dim T \geq 3$. (This will lead to a contradiction.) Let $\{R_n\}$ be an increasing sequence of positive real numbers, such that $\lim_{n\to\infty} R_n = \infty$. For every n, let T_{R_n} be the sphere in T with radius R_n (centered at the identity element). Because $\mathcal{S}^+_\Delta$ is compact and the T-orbit of Γx_0 is divergent in $\Gamma\backslash G$, we may assume that

$$(x_0 T_{R_n}) \cap (\Gamma \mathcal{S}^+_\Delta) = \emptyset \quad \text{for all } n. \tag{5.1}$$

Let

$$W_n^* = \{t \in T_{R_n} \mid x_0 t \in \Gamma Q \mathcal{S}_*\}$$

and

$$W_n = \{t \in T_{R_n} \mid x_0 t \in \Gamma Q \mathcal{S}_\alpha \cup \Gamma Q \mathcal{S}_\beta\}.$$

From Proposition 2.2, we know that for all n there exists $t_n \in T_{R_n}$, and a connected component C_n of either W_n^* or W_n, such that t_n and t_n^{-1} both belong to C_n.

CASE 1: *Assume that there are infinitely many n for which C_n is a component of W_n^*.* By passing to a subsequence, if necessary, we may assume that C_n is a connected component of W_n^* for all n. From Corollary 4.8, we see that for each n there exist $\gamma_{0n}, \gamma_n \in \Gamma$ and $q_n \in Q$, such that $x_0 t_n \in \gamma_{0n} q_n \mathcal{S}_*$, $x_0 t_n^{-1} \in \gamma_{0n}\gamma_n q_n \mathcal{S}_*$, and $q_n^{-1}\gamma_n q_n \in (UM)_{\mathbb{Q}}$. Because $\lim_{n\to\infty} \Gamma x_0 t_n = \infty$ and $\lim_{n\to\infty} \Gamma x_0 t_n^{-1} = \infty$ in $\Gamma\backslash G$, by passing to a subsequence if necessary, we must have

(1) either

$$\lim_{n\to\infty} \tilde{\alpha}(q_n^{-1}\gamma_{0n}^{-1} x_0 t_n) = \infty$$

or

$$\lim_{n\to\infty} \tilde{\beta}(q_n^{-1}\gamma_{0n}^{-1}x_0 t_n) = \infty,$$

and

(2) either

$$\lim_{n\to\infty} \tilde{\alpha}(q_n^{-1}\gamma_n^{-1}\gamma_{0n}^{-1}x_0 t_n^{-1}) = \infty$$

or

$$\lim_{n\to\infty} \tilde{\beta}(q_n^{-1}\gamma_n^{-1}\gamma_{0n}^{-1}x_0 t_n^{-1}) = \infty.$$

Since $q_n^{-1}\gamma_n q_n \in (UM)_{\mathbb{Q}}$ is sent to the identity element by both $\tilde{\alpha}$ and $\tilde{\beta}$ for all n, we have

(2′) either

$$\lim_{n\to\infty} \tilde{\alpha}(q_n^{-1}\gamma_{0n}^{-1}x_0 t_n^{-1}) = \infty$$

or

$$\lim_{n\to\infty} \tilde{\beta}(q_n^{-1}\gamma_{0n}^{-1}x_0 t_n^{-1}) = \infty.$$

Let

- $V = \bigwedge^d \mathfrak{g}$, where $d = \dim U$,
- $\rho\colon G \to \mathrm{GL}(V)$ be the d^{th} exterior power of the adjoint representation of G on V,
- $v_{\mathfrak{u}}$ be a nonzero element of $V_{\mathbb{Z}}$ in the one-dimensional subspace $\bigwedge^d \mathfrak{u}$, and
- $v_{\mathfrak{u},n} = \rho(x_0^{-1}\gamma_{0n}q_n)v_{\mathfrak{u}}$ for all n.

It is important to note that $\|v_{\mathfrak{u},n}\|$ is bounded away from 0, independent of the choice of q_n, γ_{0n} and n. (The key point is that, for each q_n, the vector $\rho(q_n)v_{\mathfrak{u}}$ is a $\mathbb{Q}$-element of V, so its $\mathbf{G}_{\mathbb{Z}}$-orbit is bounded away from 0. There are only finitely many choices of q_n, so q_n is not really an issue.)

On the other hand, for any $g \in P_{\emptyset}$, we have

$$\rho(g^{-1})v_{\mathfrak{u}} = \tilde{\alpha}(g)^{-\ell_1}\tilde{\beta}(g)^{-\ell_2}v_{\mathfrak{u}},$$

for some positive integers ℓ_1 and ℓ_2 (because the sum of the positive $\mathbb{Q}$-roots of $\mathbf{G}$ is $\ell_1\alpha + \ell_2\beta$). Therefore, from (1) and (2′), we see that

$$\lim_{n\to\infty} \rho(t_n^{-1})v_{\mathfrak{u},n} = \lim_{n\to\infty} \rho((q_n^{-1}\gamma_{0n}^{-1}x_0 t_n)^{-1})v_{\mathfrak{u}} = 0$$

and

$$\lim_{n\to\infty} \rho(t_n)v_{\mathfrak{u},n} = \lim_{n\to\infty} \rho((q_n^{-1}\gamma_{0n}^{-1}x_0 t_n^{-1})^{-1})v_{\mathfrak{u}} = 0.$$

This contradicts Lemma 3.2.

CASE 2: *Assume that there are infinitely many n for which C_n is a component of W_n.* By passing to a subsequence, if necessary, we may assume that C_n is a connected component of W_n for all n. From Lemma 4.6(2), we see that $x_0 C_n$ is contained in either $\Gamma Q \mathcal{S}_\alpha$ or $\Gamma Q \mathcal{S}_\beta$ for all n. Assume, without loss of generality, that $x_0 C_n \subset \Gamma Q \mathcal{S}_\alpha$, for all n. From Corollary 4.10, we see that for all n there exist $\gamma_{0n}, \gamma_n \in \Gamma$ and $p_n, q_n \in Q$, such that

$$x_0 t_n \in \gamma_{0n} p_n \mathcal{S}_\alpha,\ x_0 t_n^{-1} \in \gamma_{0n}\gamma_n q_n \mathcal{S}_\alpha, \text{ and } p_n^{-1}\gamma_n q_n \in h_{p_n,q_n}(U_\alpha M_\alpha)_{\mathbb{Q}}.$$

Let $\mathfrak{u}_\alpha$ be the Lie algebra of U_α, let $V_\alpha = \bigwedge^{d_\alpha} \mathfrak{g}$, where $d_\alpha = \dim \mathfrak{u}_\alpha$, and let $\rho_\alpha \colon G \to \mathrm{GL}(V_\alpha)$ be the d_α^{th} exterior power of the adjoint representation of G.

We can obtain a contradiction by arguing as in Case 1, with the representation ρ_α in the place of ρ. To see this, note that:

- For $a \in \ker\alpha$, we have $\rho_\alpha(a^{-1}) v_{\mathfrak{u}_\alpha} = \beta(a)^{-\ell} v_{\mathfrak{u}_\alpha}$, for some positive integer ℓ. Since $\rho_\alpha(UM) \subset \rho_\alpha(U_\alpha M_\alpha)$ fixes $v_{\mathfrak{u}_\alpha}$, and $\rho_\alpha(K)$ is compact, there exist constants $A, B > 0$ such that

$$A\tilde{\beta}(g)^{-\ell}\|v_{\mathfrak{u}_\alpha}\| \le \|\rho_\alpha(g^{-1})v_{\mathfrak{u}_\alpha}\| \le B\tilde{\beta}(g)^{-\ell}\|v_{\mathfrak{u}_\alpha}\| \quad \text{for } g \in \mathcal{S}_\alpha.$$

- Because $\lim_{n\to\infty}\Gamma x_0 t_n = \infty$ and $\lim_{n\to\infty}\Gamma x_0 t_n^{-1} = \infty$ in $\Gamma\backslash G$, and $\tilde{\alpha}$ is bounded on $\mathcal{S}_\alpha$, we must have
  (1) $\lim_{n\to\infty}\tilde{\beta}(p_n^{-1}\gamma_{0n}^{-1}x_0 t_n) = \infty$, and
  (2) $\lim_{n\to\infty}\tilde{\beta}(q_n^{-1}\gamma_n^{-1}\gamma_{0n}^{-1}x_0 t_n^{-1}) = \infty$.
  Therefore, letting $v_{\mathfrak{u}_\alpha,n} = \rho_\alpha(x_0^{-1}\gamma_{0n}p_n)v_{\mathfrak{u}_\alpha}$ for all n, we have

$$(1^*)\ \lim_{n\to\infty}\rho_\alpha(t_n^{-1})v_{\mathfrak{u}_\alpha,n} = 0.$$

  Because $h_{p_n,q_n} \in P_\alpha$ normalizes U_α, we have

$$\rho_\alpha(h_{p_n,q_n})v_{\mathfrak{u}_\alpha} = c_{p_n,q_n} v_{\mathfrak{u}_\alpha},$$

  for some scalar c_{p_n,q_n}. Since $(p_n^{-1}\gamma_n q_n)h_{p_n,q_n}^{-1} \in (U_\alpha M_\alpha)_{\mathbb{Q}}$ fixes $v_{\mathfrak{u}_\alpha}$, and $\{c_{p_n,q_n}\}$, being finite, is bounded away from 0, we see that

$$\begin{aligned}\rho_\alpha(t_n)v_{\mathfrak{u}_\alpha,n} &= \rho_\alpha(t_n x_0^{-1}\gamma_{0n}p_n)v_{\mathfrak{u}_\alpha}\\ &= \rho_\alpha(t_n x_0^{-1}\gamma_{0n}p_n(p_n^{-1}\gamma_n q_n)h_{p_n,q_n}{}^{-1})v_{\mathfrak{u}_\alpha}\\ &= c_{p_n,q_n}^{-1}\rho_\alpha(t_n x_0^{-1}\gamma_{0n}\gamma_n q_n)v_{\mathfrak{u}_\alpha}.\end{aligned}$$

Therefore we have

$$(2^*)\ \lim_{n\to\infty}\rho_\alpha(t_n)v_{\mathfrak{u}_\alpha,n} = \lim_{n\to\infty} c_{p_n,q_n}^{-1}\rho_\alpha(t_n x_0^{-1}\gamma_{0n}\gamma_n q_n)v_{\mathfrak{u}_\alpha} = 0.$$

This contradicts Lemma 3.2 and the proof of Theorem 1.2 is completed.

6. Results for higher $\mathbb{Q}$-rank

The proof of Theorem 1.2 generalizes to establish the following result:

6.1. THEOREM: *Suppose* $\mathbf{G}$, Γ, T, *and* x_0 *are as specified in Conjecture 1.1, and assume* $\mathbb{Q}$-rank $\mathbf{G} \geq 1$. *If the T-orbit of Γx_0 is divergent in* $\Gamma\backslash\mathbf{G}_{\mathbb{R}}$, *then* $\dim T \leq 2(\mathbb{Q}\text{-rank}\,\mathbf{G}) - 1$.

Sketch of proof: As in [L, §1 and§2], let Δ be the set of simple $\mathbb{Q}$-roots, construct a fundamental set $Q\mathcal{S}$, define the finite set $\mathcal{D}$, and choose $r > 0$, such that, for $q \in \mathcal{D}$ and $\alpha \in \Delta$, we have

$$\text{if } \tilde{\alpha} \text{ is bounded on } \mathcal{S} \cap q\mathcal{S}, \text{ then } \tilde{\alpha}(\mathcal{S} \cap q\mathcal{S}) < r.$$

Fix an increasing sequence $r = r_0 < r_0^* < r_1 < r_1^* < \cdots < r_d < r_d^*$ of real numbers. For each subset Θ of Δ, let

$$\mathcal{S}_\Theta = \{x \in \mathcal{S} \mid \tilde{\alpha}(x) < r^*_{\#\Theta},\ \forall \alpha \in \Theta\}$$

and

$$\mathcal{S}_\Theta^- = \{x \in \mathcal{S} \mid \tilde{\alpha}(x) \leq r_{\#\Theta},\ \forall \alpha \in \Theta\},$$

and choose $h_{p,q}^\Theta$ such that $p^{-1}\Gamma q \cap (P_\Theta)_{\mathbb{Q}} \subset h_{p,q}^\Theta (U_\Theta M_\Theta)_{\mathbb{Q}}$ for $p, q \in Q$. Set $d = \mathbb{Q}$-rank $\mathbf{G}$, and, for $i = 0, \ldots, d$, let

$$E_i = \bigcup_{\substack{\Theta \subset \Delta \\ \#\Theta = i}} \mathcal{S}_\Theta \quad \text{and} \quad E_i^- = \bigcup_{\substack{\Theta \subset \Delta \\ \#\Theta = i}} \mathcal{S}_\Theta^-.$$

Then $\{Q(E_0 \smallsetminus E_1^-), Q(E_1 \smallsetminus E_2^-), \ldots, Q(E_{d-1} \smallsetminus E_d^-), QE_d\}$ is an open cover of $\Gamma\backslash G$, and E_d is precompact.

For $p, q \in Q$ and $\Theta_1, \Theta_2 \subset \Delta$, let

$$\mathcal{D}_{\Theta_1,\Theta_2}^{p,q} = \{\gamma \in \Gamma \mid p\mathcal{S}_{\Theta_1} \cap \gamma q \mathcal{S}_{\Theta_2} \text{ is precompact and nonempty}\}.$$

Define

$$\mathcal{S}_\Delta^+ = \bigcup_{\substack{p,q \in Q \\ \Theta_1, \Theta_2 \subset \Delta \\ \gamma \in \mathcal{D}_{\Theta_1,\Theta_2}^{p,q}}} (\overline{p\mathcal{S}_{\Theta_1} \cap \gamma q \mathcal{S}_{\Theta_2}}).$$

Suppose $\dim T \geq 2d$. Then we may choose a $(2d-1)$-sphere T_R in T, so large that $\Gamma x_0 T_R$ is disjoint from $E_d \cup \mathcal{S}_\Delta^+$. Proposition 6.2 below implies that there exists $t \in T_R$ and a component C of some $E_{i-1} \smallsetminus E_i^-$ (with $1 \leq i \leq d$), such that $x_0 t$ and $x_0 t^{-1}$ belong to C. Since $x_0 T_R$ is disjoint from $\Gamma \mathcal{S}_\Delta^+$, then there exist $\Theta \subset \Delta$ (with $\#\Theta = i-1$), $\gamma_0, \gamma \in \Gamma$, and $p, q \in Q$, such that $x_0 t \in \gamma_0 p \mathcal{S}_\Theta$, $x_0 t^{-1} \in \gamma_0 \gamma q \mathcal{S}_\Theta$, and $p^{-1}\gamma q \in h^{\Theta}_{p,q}(U_\Theta M_\Theta)_{\mathbb{Q}}$. We obtain a contradiction as in Case 1 of §5, using $\mathfrak{u}_\Theta$ in the place of $\mathfrak{u}$. ∎

The following result is obtained from the proof of Proposition 3.1, by using the fact that any simplicial complex of dimension $d-1$ can be embedded in $\mathbb{R}^{2d-1}$.

6.2. PROPOSITION: *Suppose $n \geq 2d-1$, and that $\{V_1, V_2, \ldots, V_d\}$ is an open cover of the n-sphere S^n that consists of only d sets. Then there is a connected component C of some V_i, such that C contains two antipodal points of S^n.*

6.3. Remark: For $k \geq 1$, it is known [S, IJ] that there exist a simplicial complex Σ^k of dimension k and a continuous map $f\colon S^{2k-1} \to \Sigma^k$, such that no two antipodal points of S^{2k-1} map to the same point of Σ^k. This implies that the constant $2d-1$ in Proposition 6.2 cannot be improved to $2d-3$.

6.3. Remark: If $\mathbb{Q}$-rank $G = 2$, then the conclusion of Theorem 1.2 is stronger than that of Theorem 6.1. The improved bound in (1.2) results from the fact that if $d = 2$, then the universal cover of any $(d-1)$-dimensional simplicial complex embeds in $\mathbb{R}^2 = \mathbb{R}^{2d-2}$. (See the proof of Proposition 3.1.) When $d > 2$, there are examples of (simply connected) $(d-1)$-dimensional simplicial complexes that embed only in $\mathbb{R}^{2d-1}$, not $\mathbb{R}^{2d-2}$.

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