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Ergodic Actions of Semisimple Lie Groups on Compact Principal Bundles

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Abstract. Let $G = \mathrm{SL}(n, \mathbb{R})$ (or, more generally, let G be a connected, noncompact, simple Lie group). For any compact Lie group K, it is easy to find a compact manifold M, such that there is a volume-preserving, connection-preserving, ergodic action of G on some smooth, principal K-bundle P over M. Can M can be chosen independent of K? We show that if $M = H/\Lambda$ is a homogeneous space, and the action of G on M is by translations, then P must also be a homogeneous space H'/Λ'. Consequently, there is a strong restriction on the groups K that can arise over this particular M.

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1. Introduction

In geometric terms, the Margulis Superrigidity Theorem [4, pp. 245–246 and 332] provides an explicit description of a certain class of equivariant principal bundles. The following weak version of this result records merely the conclusion that there are only finitely many possible structure groups.

DEFINITION 1.1. A smooth action of a Lie group G on a manifold M is *ergodic* if every G-invariant Borel subset of M is either null or conull. (That is, either the set or its complement has measure 0, with respect to a Lebesgue measure on M.) A G-space M is *irreducible* if the action of each closed, noncompact, normal subgroup of G is ergodic on M.

THEOREM 1.2 (Margulis Superrigidity Theorem). *For any connected, semisimple, linear Lie group G, with $\mathbb{R}\text{-rank}\, G \geqslant 2$, there is a corresponding finite set $\mathcal{H} = \{H_1, \ldots, H_n\}$ of connected, linear Lie groups, such that if*

- *$M = \Gamma \backslash G$ is an irreducible homogeneous G-space of finite volume,*
- *H is a connected, linear Lie group,*

– *P is a principal H-bundle over M, and*
– *the action of G on M lifts to an ergodic action of G on P by bundle automorphisms,*

then H is isomorphic to one of the groups in $\mathcal{H}$.

Theorem 1.2 has been generalized to allow G-spaces that are not homogeneous, under the additional assumption that H is semisimple, with no compact factors [9, Theorem. 5.2.5, p. 98]. (We note that G, being semisimple, has only finitely many normal subgroups.)

THEOREM 1.3 (Zimmer). *If*

– *G is a connected, semisimple, linear Lie group, with $\mathbb{R}$-rank $G \geqslant 2$,*
– *M is an irreducible G-space with G-invariant finite volume,*
– *H is a connected, semisimple, linear Lie group, with no compact factors,*
– *P is a principal H-bundle over M, and*
– *the action of G on M lifts to an ergodic action of G on P by bundle automorphisms,*

then H is locally isomorphic to a normal subgroup of G.

This latter result does not address the case where the semisimple Lie group H is compact. It is perhaps surprising that the answer is completely different in this situation. Indeed, there is no restriction at all on the possible structure groups: for any compact Lie group K, it is easy to construct a G-equivariant principal K-bundle P, on which G acts irreducibly.

PROPOSITION 2.4 (1)′. *Let G be a noncompact, linear Lie group. For any compact Lie group K, there is a smooth, irreducible, volume-preserving action of G on some principal K-bundle P over some manifold M.*

For our construction in 2.4(1), the manifold M depends on the compact group K. This suggests the following problem:

PROBLEM 1.4. Let

– G be a connected, semisimple, linear Lie group, with no compact factors, such that $\mathbb{R}$-rank $G \geqslant 2$, and
– M be a smooth manifold of finite volume, on which G acts irreducibly, by volume-preserving diffeomorphisms.

Find all the connected, compact Lie groups K, for which there is a principal K-bundle P over M, such that the action of G on M lifts to an ergodic action of G on P, by bundle automorphisms. In particular, is there a choice of M for which every such group K is possible?

Our construction in 2.4(1) yields a principal bundle P with a G-invariant connection (cf. 1.7), so it is also natural to consider the following geometric version of the problem:

PROBLEM 1.5. Consider Problem 1.4, with the additional requirement that there is a G-invariant connection on the principal bundle P.

We investigate this geometric question in the special case where M is one of the known actions of G that arise from an algebraic construction. (It has been conjectured that every irreducible G-space is isomorphic to one of these known actions, modulo a nowhere-dense, G-invariant set. This may suggest that our results could be helpful in understanding the general case.)

DEFINITION 1.6. Suppose

- G is a subgroup of a Lie group H,
- Λ is a lattice in H, and
- C is a compact subgroup of H that centralizes G.

Then G acts (on the right) on the double-coset space $M = \Lambda\backslash H/C$, and we call M a *standard G-space.*

EXAMPLE 1.7. In the setting of Definition 1.6, if C acts freely on $\Lambda\backslash H$, then $P = \Lambda\backslash H$ is a principal C-bundle over M. In many cases (for example, if H is a connected, noncompact, simple group), the Mautner phenomenon [6] implies that G acts irreducibly on P.

There is a G-invariant connection on P. To see this, note that, because $GK \approx G \times K$ is reductive in H, there is an $\mathrm{Ad}_H(GK)$-invariant complement $\mathfrak{m}$ to $\mathfrak{c}$ in $\mathfrak{h}$. Then $\mathfrak{m}$ defines a GK-invariant complement to the vertical tangent space $T(P)_{\mathrm{vert}}$.

We show that if M is a standard G-space, then P must also be a standard G-space (see 3.9), so the possible choices of K are severely restricted; K must arise from purely algebraic considerations.

THEOREM 3.13′. *Suppose*

- *G is a connected, semisimple, linear Lie group, with no compact factors,*
- *$M = \Lambda\backslash H/C$ is an ergodic, standard G-space, such that H is connected and semisimple, with no compact factors,*
- *K is a connected, compact Lie group, and*
- *P is a principal K-bundle over M, such that*
    - *the action of G on M lifts to an ergodic action of G on P by bundle automorphisms, and*
    - *there is a G-invariant connection on P.*

Then there exist

- *a finite-index subgroup* Λ_0 *of* Λ,
- *a connected, compact Lie group* N,
- *a homomorphism* $\sigma\colon \Lambda_0 \to N$, *with dense image,*
- *a quotient* $\overline{C}$ *of* C°, *and*
- *a finite subgroup* F *of the center of* K,

such that $K/F \cong \overline{C} \times N$.

Combining this with the Margulis Superrigidity Theorem 1.2 yields the following result.

COROLLARY 3.14′. *In the setting of Theorem* 3.13′, *suppose that* $\mathbb{R}\text{-rank}\, H \geqslant 2$, *and that* Λ *is irreducible. Then there is a finite set* $\mathcal{K} = \{K_1, \dots, K_n\}$ *of compact, connected Lie groups, depending only on* H, *such that* K *is isomorphic to one of the groups in* $\mathcal{K}$.

2. The Measurable Category

As is well known, Theorems 1.2 and 1.3 can be restated in the language of Borel cocycles (cf. [9, Proposition 4.2.13, p. 70] and [9, Theorem 5.2.5, p. 98]).

DEFINITION 2.1. Suppose a Lie group G acts measurably on a Borel space M, with a quasi-invariant measure μ, and H is a (second countable) locally compact group.

- A Borel measurable function $\alpha\colon M \times G \to H$ is a *Borel cocycle* if, for each $g, h \in G$, we have $\alpha(x, gh) = \alpha(x, g)\,\alpha(xg, h)$ for a.e. $x \in M$.
- A Borel cocycle α is *strict* if the equality holds for every x, instead of only for almost every x.
- If $\alpha\colon M \times G \to H$ is a strict Borel cocycle, then the *skew-product action* $M \times_\alpha H$ of G is the action of G on $M \times H$, given by $(m, h)g = (mg, h\,\alpha(m, g))$.

Any Borel cocycle is equal, almost everywhere, to a strict Borel cocycle [9, Theorem B.9, p. 200], so, abusing terminology, we will speak of the skew-product action $M \times_\alpha H$ even if α is not strict.

The following statement strengthens Theorem 1.2 by incorporating the additional conclusion that the subgroup H must be compact.

THEOREM 1.2′. *For any connected, semisimple, linear Lie group* G, *with* $\mathbb{R}\text{-rank}\, G \geqslant 2$, *there is a corresponding finite set* $\mathcal{H} = \{H_1, \dots, H_n\}$ *of connected, compact Lie groups, such that if*

- H *is a connected, linear, Lie group,*
- Γ *is an irreducible lattice in* G,

– *$\alpha\colon (\Gamma\backslash G) \times G \to H$ is a Borel cocycle, and*
– *the skew-product action $(\Gamma\backslash G) \times_\alpha H$ is ergodic,*

then H is isomorphic to one of the groups in $\mathcal{H}$.

THEOREM 1.3′ (Cocycle Superrigidity Theorem). *Let*

– *G be a connected, semisimple, linear Lie group, with no compact factors, such that $\mathbb{R}$-rank $G \geqslant 2$,*
– *M be an irreducible, ergodic G-space with finite invariant measure,*
– *H be a connected, semisimple, linear Lie group, with no compact factors, and*
– *$\alpha\colon M \times G \to H$ be a Borel cocycle.*

If the skew-product action $M \times_\alpha H$ is ergodic, then H is locally isomorphic to a normal subgroup of G.

Recall that if P is a principal K-bundle over a manifold M, then any action of G on P by bundle automorphisms yields a Borel cocycle $\alpha\colon M \times G \to K$, such that P is G-equivariantly measurably isomorphic to the skew product $M \times_\alpha K$ (cf. [9, pp. 66–67]). This suggests the following measurable version of Problem 1.4.

PROBLEM 2.2. Let

– G be a connected, semisimple, linear Lie group, with no compact factors, such that $\mathbb{R}$-rank $G \geqslant 2$,
– M be a smooth manifold of finite volume, on which G acts irreducibly, by volume-preserving diffeomorphisms, and
– K be a connected, compact Lie group.

Describe the Borel cocycles $\alpha\colon M \times G \to K$, such that the skew-product action $M \times_\alpha K$ is ergodic. In particular, are there choices of G, M, and K for which no such cocycle exists? (We assume that every G-orbit on M is a null set, for otherwise Theorem 1.2′ applies.)

PROPOSITION 2.3. *For any noncompact, linear Lie group G, and any compact (second countable) group K, there is an irreducible, ergodic G-space M with finite invariant measure, and a Borel cocycle $\alpha\colon M \times G \to K$, such that the skew-product $M \times_\alpha K$ is ergodic (and irreducible).*

Proof. Embed G in some $\mathrm{SL}(n, \mathbb{R})$. For each natural number k, let $H_k = \mathrm{SL}(n+k, \mathbb{R})$ and let Λ_k be a (cocompact) torsion-free lattice in H_k. Let

$$M_0 = \frac{H_2}{\Lambda_2} \times \frac{H_3}{\Lambda_3} \times \frac{H_4}{\Lambda_4} \times \cdots$$

and

$$K_0 = \mathrm{SO}(2) \times \mathrm{SO}(3) \times \mathrm{SO}(4) \times \cdots.$$

The group G acts diagonally on M_0. From vanishing of matrix coefficients [9, Theorem 2.2.20, p. 23], we know that the diagonal embedding of G is mixing on each finite subproduct of M_0, so (by approximating arbitrary sets by finite unions of cylinders) we conclude that G is mixing on the entire product M_0. In particular, every closed, noncompact subgroup of G is ergodic on M_0, so M_0 is irreducible.

By construction, H_k contains $G \times SO(k)$, so there is a free action of K_0 on M_0 that centralizes the diagonal action of G. Then, because K is isomorphic to a closed subgroup of K_0, we know that there is a free action of K on M_0 that centralizes the diagonal action of G. Let $M = M_0/K$, so M_0 is a principal K-bundle over M. ∎

COROLLARY 2.4.

(1) *If K is a Lie group, then the G-space M can be taken to be a smooth, compact manifold, and the skew product $M \times_\alpha K$ can be realized as a principal K-bundle over M.*
(2) *If K is restricted to the class of Lie groups, then there is a G-space M_1 that works for all K simultaneously. (That is, there is no need to vary M as K ranges over the set of compact Lie groups.)*
(3) *Similarly, if K is restricted to the class of compact, abelian groups that are not necessarily Lie, then there is a G-space M_2 that works for all K simultaneously.*

Proof. (1) Because $K \subset SO(k)$, for some k, we may use $\Lambda_k \backslash H_k$, which is a smooth manifold, instead of M_0, in the construction.

(2) Let K_1 be the direct product of one representative from each isomorphism class of compact Lie groups. There are only countably many compact Lie groups, up to isomorphism [1, Corollary 10.13], so K_1 is compact and second countable. From the proposition, there is an ergodic G-space M_1 with finite invariant measure and a K_1-valued cocycle α_1 with ergodic skew product. Now, for any K, we simply let α be the composition of α_1 with the projection from K_1 to K.

(3) By the argument of (2), it suffices to show that there is a compact abelian group A_0, such that every compact abelian group is a quotient of A_0. To see this, let A be the direct sum of countably many copies of $\mathbb{Q}$ and countably many copies of $\mathbb{Q}/\mathbb{Z}$, then let A_0 be the Pontryagin dual of A. Every countable Abelian group is isomorphic to a subgroup of A (because every countable Abelian group is isomorphic to a subgroup of a countable, divisible, Abelian group [2, Theorem 24.1, p. 106] and every countable, divisible, Abelian group is isomorphic to a subgroup of A [2, Theorem 23.1, p. 104]), so, by duality, every compact Abelian group is isomorphic to a quotient of A_0. ∎

The following observation is well known, but does not seem to have previously appeared in print.

Remark 2.5. Assume G is noncompact and has Kazhdan's property T (for example, let $G = \mathrm{SL}(3, \mathbb{R})$ [9, Theorem 7.4.2, p. 146]), and let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the circle group. From Proposition 2.3, we know there is an ergodic G-space M, with finite invariant measure, and a Borel cocycle $\alpha: M \times G \to \mathbb{T}$, such that $M \times_\alpha \mathbb{T}$ is ergodic. Then α is not cohomologous to any cocycle β, such that $\beta(M \times G)$ is countable. To see this, we argue by contradiction: let A be the subgroup of $\mathbb{T}$ generated by $\alpha(M \times G)$, and suppose, after replacing α with a cohomologous cocycle, that A is countable. Then we may think of α as a cocycle into A (with the discrete topology on A). Because G has Kazhdan's property T, and A is Abelian, we conclude that α is cohomologous to a cocycle whose values lie in a compact (thus, finite) subgroup of A [9, Theorem 9.1.1, p. 162]. Thus, we may assume $\alpha(M \times G)$ is finite. But then $M \times_\alpha \mathbb{T}$ is clearly not ergodic. This contradicts the choice of α.

3. The Geometric Category

THEOREM 3.1. *Let*

- *H be a connected Lie group,*
- *$M = \Lambda\backslash H$, for some lattice Λ in H, such that H acts faithfully on M,*
- *G be a connected, semisimple Lie subgroup of H, with no compact factors,*
- *K be a compact Lie group,*
- *$E \to M$ be a smooth principal K-bundle, such that the action of G on M lifts to a well-defined (faithful) action of a cover G' of G by bundle automorphisms of E.*

Assume that

- *G is ergodic on M, and*
- *G' preserves a connection on E.*

Then there exist

- *a closed subgroup N of K,*
- *a G'-invariant principal N-subbundle E' of E, and*
- *a Lie group H', with only finitely many connected components,*

such that

(1) *H' is a transitive group of diffeomorphisms of E',*
(2) *H' contains N as a normal subgroup,*
(3) *H'/N is isomorphic to H,*
(4) *the action induced by H' on $E'/N = M$ is the action of H on M,*
(5) *H' contains G', and*
(6) *G' is ergodic on E'.*

Therefore, there is a lattice Λ' in H', such that

(a) *E' is G'-equivariantly diffeomorphic to $\Lambda'\backslash H'$, and*
(b) *M is G-equivariantly diffeomorphic to $\Lambda'\backslash H'/N$.*

Remark 3.2. Because K centralizes G', we see that G' is ergodic on $E'k$, for every $k \in K$. Therefore, $\{E'k \mid k \in K\}$ is an ergodic decomposition for the G'-action on E. In particular, if G' is ergodic on E, then $E' = E$, so $N = K$.

Remark 3.3. The proof shows that if K is connected and G' is ergodic on E, then H' may be taken to be connected. In general, H' is constructed so that $H' = (H')^{\circ}N$.

Proof. We begin by establishing some notation.

- Let ω be the connection form of some G'-invariant connection [3, pp. 63–64].
- Let $\Omega = D\omega$ be the curvature form of the connection [3, pp. 77].
- We view $\mathfrak{h}$ as the Lie algebra of left-invariant vector fields on H, so each element of $\mathfrak{h}$ is well-defined as a vector field on $\Lambda\backslash H = M$.
- For each $X \in \mathfrak{h}$, we use $\bar{X}$ to denote the lift of X to a horizontal vector field on E.
- For $Z \in \mathfrak{k}$, let $\check{Z}$ be the corresponding vertical vector field on E induced by the action of K (so $\omega(\check{Z}) = Z$).
- For any $X \in \mathfrak{g}$, let X' be the corresponding vector field on E induced by the action of G'.

By definition, Ω is a horizontal 2-form on E taking values in $\mathfrak{k}$. For each $e \in E$, the connection provides an identification of the horizontal part of the tangent space T_eE with $\mathfrak{h}$; therefore, we may think of Ω as a map $\hat{\Omega}: E \to \mathrm{Hom}_{\mathbb{R}}(\mathfrak{h} \wedge \mathfrak{h}, \mathfrak{k})$. From [3, Proposition II.5.1(c), p. 76 and comments on p. 77], we have

$$\hat{\Omega}_{ek} = (\mathrm{Ad}_K k^{-1}) \circ \hat{\Omega}_e \text{ for every } e \in E \quad \text{and} \quad k \in K. \tag{3.4}$$

Step 1. We have

(1) $\hat{\Omega}_{eg} = \hat{\Omega}_e$ for every $e \in E$ and $g \in G'$; and
(2) there is some $\phi \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{h} \wedge \mathfrak{h}, \mathfrak{k})$, such that $\hat{\Omega}_e \in \{(\mathrm{Ad}_K k) \circ \phi | k \in K\}$, for every $e \in E$.

For any $X \in \mathfrak{h}$ and $g, a \in H$, and defining $R_g: H \to H$ by $R_g(h) = hg$, the left-invariance of X implies $\mathrm{d}(R_g)_a(X_a) = \big((\mathrm{Ad}_H g^{-1})X\big)_{ga}$ [3, Proposition I.5.1, p. 51]; therefore, for $g \in G'$ and $e \in E$,

$$\text{the horizontal part of } \mathrm{d}g_e(\bar{X}_e) \text{ is } \big(\overline{(\mathrm{Ad}_{H} g^{-1})X}\big)_{eg}. \tag{3.5}$$

Then, because the connection is G'-invariant (and G' commutes with the K-action on E), we have

$$\begin{aligned}
\hat{\Omega}_e(X, Y) &= \Omega_e(\bar{X}_e, \bar{Y}_e) && \text{(definition of } \hat{\Omega}) \\
&= \Omega_{eg}\big(dg_e(\bar{X}_e), dg_e(\bar{Y}_e)\big) && \text{[3, pp. 79–80]} \\
&= \Omega_{eg}\Big(\big(\overline{(\mathrm{Ad}_H g^{-1})X}\big)_{eg}, \big(\overline{(\mathrm{Ad}_H g^{-1})Y}\big)_{eg}\Big) && \\
& && \text{((3.5) and } \Omega \text{ is horizontal)} \\
&= \hat{\Omega}_{eg}\big((\mathrm{Ad}_H g^{-1})X, (\mathrm{Ad}_H g^{-1})Y\big) && \text{(definition of } \hat{\Omega}\text{),}
\end{aligned} \tag{3.6}$$

so $\hat{\Omega}$ is G'-equivariant. Then, since the action of G' on E preserves a probability measure, but the action on $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{h} \wedge \mathfrak{h}, \mathfrak{k})$ is algebraic, the Borel Density Theorem implies that $\hat{\Omega}$ is constant on almost every ergodic component of the G'-action on E (see [5, Corollary 3.1]). Because almost every G-orbit on M is dense, this implies that we may choose some $e_0 \in E$, such that $\hat{\Omega}$ is constant on $e_0 G'$, and the projection of $e_0 G'$ to M is dense.

Let E' be the closure of $e_0 G'$, and let $\phi = \hat{\Omega}_{e_0}$. Because $\hat{\Omega}$ is continuous, we know that $\hat{\Omega}$ is constant on E'. Because K is compact, the projection $\phi\colon E \to M$ is a proper map, so $\pi(E')$ is closed in M. Since it is also dense, we conclude that $\pi(E') = M$; thus, $E = E'K$. Now, for any $e \in E'$ and $k \in K$, we have

$$\begin{aligned} \hat{\Omega}_{ek} &= (\mathrm{Ad}_K k^{-1}) \circ \hat{\Omega}_e && \text{(see (3.4))} \\ &= (\mathrm{Ad}_K k^{-1}) \circ \hat{\Omega}_{e_0} && (\hat{\Omega} \text{ is constant on } E') \\ &= (\mathrm{Ad}_K k^{-1}) \circ \phi && (\text{definition of } \phi). \end{aligned}$$

This implies that

(1) $\hat{\Omega}$ is constant on $E'k$, for each $k \in K$, so $\hat{\Omega}$ is G'-invariant, and
(2) $\hat{\Omega} \bullet \{(\mathrm{Ad}_K k) \circ \phi \mid k \in K\}$, for every $e \in E'K = E$.

Step 2. We may assume that $\hat{\Omega}$ is constant. Let

- ϕ be as in Step 1,
- $E' = \{e \in E \mid \hat{\Omega}_e = \phi\}$, and
- $N = \{k \in K \mid (\mathrm{Ad}_K k) \circ \phi = \phi\}$.

The Implicit Function Theorem (together with (3.4) and Conclusion (2) of Step 1) implies that E' is a smooth submanifold of E. Then, from (3.4), we see that E' is a principal N-subbundle of E. Because $\hat{\Omega}$ is G'-invariant, this subbundle is G'-invariant.

Because N is compact, we know that $\mathrm{Ad}_K N$ is completely reducible, so there is an $(\mathrm{Ad}_K N)$-equivariant projection $p\colon \mathfrak{k} \to \mathfrak{n}$; let $\omega' = p \circ \omega$. Then ω' is a $\mathfrak{n}$-valued 1-form. Because p is $(\mathrm{Ad}_K N)$-equivariant, we have $R_k^* \omega' = (\mathrm{Ad}_N k^{-1}) \circ \omega'$ for all $k \bullet N$, so [3, Proposition II.1.1, p. 64] implies that ω' is the connection form of a connection on E'. Since the connection corresponding to ω is G'-invariant, and G' centralizes K, it is easy to see that the connection corresponding to ω' is also G'-invariant.

If $N \neq K$, then, by replacing E with the subbundle E', we may replace K with a smaller subgroup. By the descending chain condition on closed subgroups of K, this cannot continue indefinitely.

Thus, we may assume $N = K$. Then $E' = E$, so $\hat{\Omega}$ is constant on E, as desired.

Step 3. The vector space $\overline{\mathfrak{h}} + \check{\mathfrak{k}}$ is a Lie algebra. Note that $[\check{\mathfrak{k}}, \overline{\mathfrak{h}}] = 0$ (cf. [3, Proposition II.1.2, p. 65]), so we need only show $[\overline{\mathfrak{h}}, \overline{\mathfrak{h}}] \subset \overline{\mathfrak{h}} + \check{\mathfrak{k}}$. Fix $X, Y \in \mathfrak{h}$, and let $Z = -2\hat{\Omega}(X, Y)$. Then, from [3, Corollary II.5.3, p. 78], we have

$$\omega([\overline{X}, \overline{Y}]) = -2\Omega(\overline{X}, \overline{Y}) = -2\hat{\Omega}(X, Y) = Z = \omega(\check{Z}),$$

so we see that

$$[\bar{X}, \bar{Y}] = \overline{[X, Y]} + \check{Z} \in \bar{\mathfrak{h}} + \check{\mathfrak{k}},$$

as desired.

Step 4. We have $\mathfrak{g}' \subset \bar{\mathfrak{h}}$. For $X, Y, Z \in \mathfrak{g}$, and letting $\sum$ denote summation over the set of cyclic permutations of (X, Y, Z), we have the following well-known calculation:

$$\begin{aligned}
-\sum \hat{\Omega}([X, Y], Z) &= -\sum \Omega(\overline{[X, Y]}, \bar{Z}) && \text{(definition of } \hat{\Omega}\text{)}\\
&= -\sum \Omega([\bar{X}, \bar{Y}], \bar{Z}) && (\Omega \text{ is horizontal})\\
&= -\sum \Omega([\bar{X}, \bar{Y}], \bar{Z})\\
&\quad + \sum \bar{X}(\Omega(\bar{Y}, \bar{Z}))\\
& && (\Omega(\bar{Y}, \bar{Z}) = \hat{\Omega}(Y, Z) \text{ is constant})\\
&= 3\, d\Omega(\bar{X}, \bar{Y}, \bar{Z}) && \text{[3, Proposition II.3.11, p. 36]}\\
&= 3\, D\Omega(\bar{X}, \bar{Y}, \bar{Z}) && (\bar{X}, \bar{Y}, \text{ and } \bar{Z} \text{ are horizontal})\\
&= 0 && \text{(Bianchi Id. [3, p. 78]).}
\end{aligned}$$

Thus, as is well known, $\hat{\Omega}|_{\mathfrak{g} \wedge \mathfrak{g}}$ is a 2-cocycle for the Lie algebra cohomology of $\mathfrak{g}$ (with coefficients in the module $\mathfrak{k}$ with trivial $\mathfrak{g}$-action) [8, (3.12.5), p. 220]. From Whitehead's Lemma [8, Theorem 3.12.1, p. 220], we know that $H^2(\mathfrak{g}; \mathfrak{k}) = 0$, so there is some $\sigma \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathfrak{k})$, such that

$$\hat{\Omega}(X, Y) = \sigma([X, Y]) \quad \text{for all } X, Y \in \mathfrak{g}.$$

For $g \in G$ and $X, Y \in \mathfrak{g}$, we have

$$\begin{aligned}
\sigma([X, Y]) &= \hat{\Omega}(X, Y) && \text{(definition of } \sigma\text{)}\\
&= \hat{\Omega}((\mathrm{Ad}_H g^{-1})X, (\mathrm{Ad}_H g^{-1})Y) && \text{(3.6), } \hat{\Omega} \text{ is constant)}\\
&= \sigma([(\mathrm{Ad}_H g^{-1})X, (\mathrm{Ad}_H g^{-1})Y]) && \text{(definition of } \sigma\text{)}\\
&= \sigma((\mathrm{Ad}_H g^{-1})[X, Y]) && (\mathrm{Ad}_H g^{-1} \text{ is an automorphism of } \mathfrak{h}),
\end{aligned}$$

Then, because $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, we conclude that $\sigma((\mathrm{Ad} g)X - X) = 0$, for every $g \in G$ and $X \in \mathfrak{g}$. Because G is semisimple, we have $[G, \mathfrak{g}] = \mathfrak{g}$, so we conclude that $\sigma(\mathfrak{g}) = 0$. Combining this with the definitions of σ and $\hat{\Omega}$, and the fact that the form Ω is horizontal, we have

$$0 = \sigma([X, Y]) = \hat{\Omega}(X, Y) = \Omega(\bar{X}, \bar{Y}) = \Omega(X', Y'), \tag{3.7}$$

for every $X, Y \in \mathfrak{g}$.

For any fixed $e_0 \in E$, define $\tau\colon \mathfrak{g} \to \mathfrak{k}$ by $\tau(X) = \omega(X'(e_0))$. Then, because

$$2\Omega(X', Y') = [\tau(X), \tau(Y)] - \tau([X, Y])$$

[3, Proposition II.11.4, p. 106], we conclude, from (3.7), that $\tau\colon \mathfrak{g} \to \mathfrak{k}$ is a homomorphism. Since G has no compact factors, this implies $\tau = 0$. Because e_0 is an arbitrary element of E, we conclude that $\omega(\mathfrak{g}') = 0$, so every vector field in $\mathfrak{g}'$ is horizontal. Therefore $\mathfrak{g}' = \bar{\mathfrak{g}} \subset \bar{\mathfrak{h}}$.

Step 5. *Completion of the proof.* Let H_0' be the connected Lie group of diffeomorphisms of E corresponding to the Lie algebra $\overline{\mathfrak{h}} + \check{\mathfrak{k}}$. Because K centralizes $\overline{\mathfrak{h}}$ [3, Proposition II.1.2, p. 65] and (obviously) K is normalized by K, we see that K is normalized by H_0'. Let $H' = H_0'K$ (so H_0' is the identity component of H', and H_0' is a finite-index subgroup of H'); then K is a normal subgroup of H'. Furthermore, because $\mathfrak{g}' \subset \overline{\mathfrak{h}} \subset \mathfrak{h}'$, we have $G' \subset H'$.

Because $\mathfrak{h}' = \mathfrak{h}'_0 = \overline{\mathfrak{h}} + \check{\mathfrak{k}}$, we see that H' is transitive on E, with discrete stabilizer, so there is a discrete subgroup Λ' of H', such that E is H'-equivariantly diffeomorphic to $\Lambda' \backslash H'$. Then, because $G' \subset H'$, we know that E is G'-equivariantly diffeomorphic to $\Lambda' \backslash H'$. Then, because

$$\mathfrak{h}'/\check{\mathfrak{k}} = (\overline{\mathfrak{h}} + \check{\mathfrak{k}})/\check{\mathfrak{k}} \cong \mathfrak{h}$$

and $H' = H_0'K$, we see that the action induced by H' on $\Lambda' \backslash H'/K \cong E/K = M$ is the action of H on M. This implies that H'/K is a cover of H. Also, because there is an H-invariant probability measure on $M = \Lambda \backslash H$, and K is compact, this implies that there is an H'-invariant probability measure on $\Lambda' \backslash H'$; thus, Λ' is a lattice in H'.

Let L be the smallest normal subgroup of H_0' that contains G'. (Note that L is also normal in H' (because $H' = H_0'K$ and K centralizes G'), so $\Lambda' L$ is a subgroup of H'.) Let

- H'' be the identity component of the closure of $\Lambda' L$,
- $N' = K \cap \Lambda' H''$, and
- $E'' = \Lambda' \backslash \Lambda' H'' \subset E$.

Note that Λ' normalizes H'', so $\Lambda' H''$ is a (closed) subgroup of H'.

Because G is ergodic on M, we may assume that $\Lambda' G' K$ is dense in H'. Then, because $\Lambda' H''$ is closed and contains $\Lambda' G'$, we conclude that $\Lambda' H'' K = H'$. This means $E'' K = E$, so E'' projects onto all of M.

We claim that E'' is a principal N'-bundle over M. It is clear that E'' is N'-invariant, so E'' is a principal N'-bundle over E''/N'. Thus, we need only show that the natural map $E''/N' \to E/K = M$ is injective. For $\lambda_1, \lambda_2 \in \Lambda'$, $h_1, h_2 \in H''$, and $k \in K$, such that $\lambda_1 h_1 k = \lambda_2 h_2$, we have

$$k = (\lambda_1 h_1)^{-1}(\lambda_2 h_2) \in \Lambda' H'',$$

so $k \in N'$, as desired.

Arguing as in the last three paragraphs of Step 2, we see that we may assume $N' = K$. Then $E'' = E$, so $\Lambda' L$ is dense in $\Lambda' \backslash H'$. From the Mautner Phenomenon [6], we conclude that G' is ergodic on $\Lambda' \backslash H' = E$.

All that remains is to show that H'/K acts faithfully on M. (This implies that H'/K is isomorphic to H.) Thus, we wish to show that K is the kernel of the action of H' on $\Lambda' \backslash H'/K$. Now H' is transitive on $\Lambda' \backslash H'/K$, with stabilizer $\Lambda' K$. Thus, the kernel of the action is a (normal) subgroup of the form LK, where L is a subgroup of Λ'. (We wish to show L is trivial.) Let $\lambda \in L$. Then, because L is discrete and G' is connected, we must have $\lambda^{G'} \subset \lambda K$. Since K is compact, but G' has no compact factors, we conclude that $\lambda^{G'} = \lambda$. Therefore $L^{G'} = L$, so $L^{G'\Lambda'} = L^{\Lambda'} \subset \Lambda'$. Because G' is ergodic on $E = \Lambda' \backslash H'$, we may assume that $G'\Lambda'$ is dense in H'. So $L^{H'} \subset \Lambda'$. Since H' is faithful on $\Lambda' \backslash H'$, we know Λ' does not contain any nontrivial normal subgroups of H'. Therefore $L^{H'}$ must be trivial, so L is trivial, as desired. □

Remark 3.8. If G' is ergodic on E, and the center of K is discrete, then combining Step 2 with (3.4) yields the conclusion that $\hat{\Omega} = 0$; thus, under these assumptions, every G'-invariant connection on E is flat.

The following corollary generalizes Theorem 3.1, by allowing M to be a double-coset space $\Lambda \backslash H / C$, instead of requiring it to be a homogeneous space $\Lambda \backslash H$. However, we add the additional assumption that G is ergodic on the bundle P. See Remark 3.12 for a discussion of some other less important assumptions in the statement of this more general result.

COROLLARY 3.9. *Let*

- *H be a connected Lie group,*
- *Λ be a torsion-free lattice in H, such that H acts faithfully on $\Lambda \backslash H$,*
- *C be a compact subgroup of H,*
- *$M = \Lambda \backslash H / C$,*
- *G be a connected, semisimple Lie subgroup of H, with no compact factors,*
- *K be a connected, compact Lie group,*
- *$P \to M$ be a smooth principal K-bundle, such that the action of G on M lifts to a well-defined (faithful) action of a cover G' of G by bundle automorphisms of P.*

Assume that

- *G is ergodic on $\Lambda \backslash H$,*
- *G is ergodic on P, and*
- *G' preserves a connection on P.*

Then there exist

- *a Lie group H', with only finitely many connected components,*
- *a lattice Λ' in H',*
- *a compact subgroup C' of H',*
- *a G'-equivariant diffeomorphism $\phi: \Lambda' \backslash H' / C' \to P$, and*
- *a continuous, surjective homomorphism $\rho: H' \to H$, with compact kernel,*

such that, letting $K' = \rho^{-1}(C)$, we have

(1) *H' contains G', and G' acts ergodically on $\Lambda' \backslash H'$,*
(2) *G' centralizes K', and C' is a normal subgroup of K',*

(3) *ϕ conjugates the action of K'/C' on $\Lambda'\backslash H'/C'$ to the action of K on P, and*
(4) *ϕ factors through to a diffeomorphism that conjugates the action of G' on $\Lambda'\backslash H'/K'$ to the action of G on $M = \Lambda\backslash H/K$.*

Proof. Let

- $\sigma: \Lambda\backslash H \to \lambda\backslash H/C$ be the natural quotient map;
- $\zeta: P \to \Lambda\backslash H/C$ be the bundle map; and
- $E = \{(x, p) \in (\Lambda\backslash H) \times P \mid \sigma(x) = \zeta(p)\}$.

Then E is the principal K-bundle over $\Lambda\backslash H$ obtained as the pullback of P. Note that

- G' acts (diagonally) on E, via $(x, p)g = (xg, pg)$, and
- K acts on E, via $(x, p)k = (x, pk)$.

Warning: G' may not be ergodic on E (see 3.15).

Step 1. G' preserves a connection on E. Let ω be the connection form of a G'-invariant connection on P, and define $\omega^E_{(x,p)}(v, w) = \omega_p(w) \in \mathfrak{k}$ for $e = (x, p) \in E$ and

$$(v, w) \in T_{(x,p)}(E) \subset T_x(\Lambda\backslash H) \oplus T_p(P).$$

Because ω is a connection form, it is easy to see, from the characterization of connection forms [3, Proposition 2.1.1, p. 64], that ω^E is the connection form of a connection on E. Because ω is G'-invariant, it is clear that ω^E is G'-equivariant.

Step 2. Let

- N, E', H', and Λ' be as in Theorem 3.1,
- $\psi: \Lambda'\backslash H' \to E'$ be an H'-equivariant diffeomorphism, so $\psi: \Lambda'\backslash H' \hookrightarrow E$,
- $\rho: H' \to H$ be the surjective homomorphism, with kernel N, that results from 3.1(4), and
- $\zeta': \Lambda'\backslash H' \to \Lambda\backslash H$ be the affine map induced by ρ.

$$\begin{array}{ccccccccc} H' & \longrightarrow & \Lambda'\backslash H' & \xrightarrow{\psi} & E & \xrightarrow{\pi_2} & P & & \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\zeta'} & & \downarrow{\scriptstyle\pi_1} & & \downarrow{\scriptstyle\zeta} & & \\ H & \longrightarrow & \Lambda\backslash H & = & \Lambda\backslash H & \xrightarrow{\sigma} & \Lambda\backslash H/C & = & M \end{array} \tag{3.10}$$

Step 3. $K' = \rho^{-1}(C)$ is compact, and G' centralizes K'. Because G normalizes C, we know, by pulling back to H', that G' normalizes K'. Furthermore, because both N and $K'/N \cong C$ are compact, we know that K' is compact. Since G' has no compact factors, and normalizes K', we conclude that G' centralizes K'.

Step 4. *ψ factors through to a diffeomorphism that conjugates the action of G' on $\Lambda'\backslash H'/K'$ to the action of G on $\Lambda\backslash H/C$.* From the definition of ρ, we see that the

G'-equivariant map ψ induces a diffeomorphism that conjugates the action of K' on $\Lambda'\backslash H'/N$ to the action of $\rho(K') = C$ on $\Lambda\backslash H$. Thus, it factors through to a G'-equivariant diffeomorphism between the orbit spaces of these actions.

Step 5. There is a closed subgroup C' of K', such that for $e_1, e_2 \in E'$, we have $\pi_2(e_1) = \pi_2(e_2)$ if and only if $e_1 \in e_2C'$; hence $\pi_2 \circ \psi$ factors through to a G'-equivariant diffeomorphism $\phi\colon \Lambda'\backslash H'/C' \to P$. For $e \in E'$, let

$$\phi(e) = \{\, c \in K' \mid \pi_2(ec) = \pi_2(e)\}.$$

Because G' centralizes K', and because, from its definition, π_2 is obviously G'-equivariant, we have $\phi(eg) = \phi(e)$ for $e \in E'$ and $g \in G'$. Then, because G' is ergodic on E', we conclude that ϕ is essentially constant. By continuity, it must be constant: let $C' = \phi(e)$ for any $e \in E'$.

From the definition of ϕ, we see that if $e_1 \in e_2C'$, then $\pi_2(e_1) = \pi_2(e_2)$. Conversely, suppose $\pi_2(e_1) = \pi_2(e_2)$. Write $e_1 = \psi(\Lambda'h_1)$ and $e_2 = \psi(\Lambda'h_2)$, for some $h_1, h_2 \in H'$. Because $\zeta\big(\pi_2(e_1)\big) = \zeta\big(\pi_2(e_3)\big)$, we see, from the commutative diagram (3.10), that $\zeta'(\Lambda'h_1) \in \zeta'(\Lambda'h_2)C$. Thus, from the definitions of ζ' and K', we conclude that

$$\Lambda'h_1 \in \Lambda'h_2 \cdot \rho^{-1}(C) = \Lambda'h_2K'.$$

So

$$e_1 = \psi(\Lambda'h_1) \in \psi(\Lambda'h_2K') = \psi(\Lambda'h_2)K' = e_2K'.$$

Then, because $\pi_2(e_1) = \pi_2(e_2)$, we conclude, from the definition of C', that $e_1 \in e_2C'$.

For $c_1, c_2 \in C'$, and any $e \in E'$, we have

$$\pi_2(ec_1c_2) = \pi_2(ec_1) = \pi_2(e),$$

so $c_1c_2 \in C'$. Therefore, C' is a subgroup of K'. Because H' acts continuously, and π_2 is continuous, we know that C' is closed.

Because $\pi_2 \circ \psi$ is G'-equivariant, and G' is ergodic on P, we see that $\pi_2 \circ \psi$ is surjective. Hence

$$P = \pi_2(E') \approx E'/C' \approx \Lambda'\backslash H'/C'.$$

Step 6. *C' is normal in K', with $K'/C' \cong K$, and ϕ conjugates the action of K'/C' on $\Lambda'\backslash H'/C'$ to the action of K on P.* Fix some $e \in E'$. From the commutative diagram (3.10), and the definition of K', we see that $\pi_2(eK')$ is a fiber of ζ; hence, Step 5 implies that $C'\backslash K'$ is diffeomorphic to a fiber of ζ. Because K is connected, we know that the fiber is connected, so we conclude that $C'\backslash K'$ is connected. Therefore $K' = (K')^\circ C'$.

Let $X \in \mathfrak{k}'$, and let X' be the corresponding vector field on $\Lambda'\backslash H'$ induced by the action of K' on $\Lambda'\backslash H'$. Any element Z of $\mathfrak{k}$ induces a vertical vector field $\check{Z}$ on P, and $\check{\mathfrak{k}}$ constitutes the entire vertical tangent space at each point. Thus, for any fixed $x_0 \in \Lambda'\backslash H'$, there is some $Z \in \mathfrak{k}$, such that we have $d(\pi_2 \circ \psi)_{x_0}(X') = \check{Z}_{\pi_2(\psi(x_0))}$. Now, each of the maps $x \mapsto d(\pi_2 \circ \psi)_x(X')$ and $x \mapsto \check{Z}_{\pi_2(\psi(x_0))}$ is G'-equivariant (because G' centralizes both K' and K), so these maps are equal on the orbit x_0G'. We may choose x_0 so that this orbit is dense, and then we conclude, by continuity, that

$$d(\pi_2 \circ \psi)_x(X') = \check{Z}_{\pi_2(\psi(x_0))} \quad \text{for all } x \in \Lambda' \backslash H'. \tag{3.11}$$

This implies that X' factors through to a well-defined vector field on $\Lambda'\backslash H'/C'$. Hence $(\mathrm{Ad}\, c)X \in X + \mathfrak{c}'$, for all $c \in C'$. Because X is an arbitrary element of $\mathfrak{k}'$, this implies that $(K')^\circ$ normalizes C'. Because $K' = (K')^\circ C'$, we conclude that K' normalizes C'. Furthermore, because $d(\pi_2 \circ \psi)$ maps $\mathfrak{k}'$ to $\mathfrak{k}$ (see 3.11), we know that $\pi_2 \circ \psi$ conjugates the action of K' on $\Lambda'\backslash H'$ to the action of K on P. Hence, ϕ conjugates the action of K'/C' on $\Lambda'\backslash H'/C'$ to the action of K on P. ■

Remark 3.12. Some of the technical assumptions in Corollary 3.9 are not very important; they can be satisfied by passing to a finite cover, or making other similar minor adjustments.

(1) One might assume only that M is connected, rather than that H is connected. Then $\Lambda_0\backslash H^\circ/C_0$ is a finite cover of M, where $\Lambda_0 = \Lambda \cap H^\circ$ and $C_0 = C \cap H^\circ$.
(2) The assumption that H acts faithfully on $\Lambda\backslash H$ is only a convenience; one could always mod out the kernel of this action.
(3) If one does not assume that Λ is torsion free, then Selberg's Lemma [7, Corollary 6.13, p. 95] implies that Λ has a torsion-free subgroup Λ_0 of finite index. The space $\Lambda_0\backslash H/C$ is a finite cover of M.
(4) If one does not assume that K is connected, then P/K° is a finite cover of M.
(5) If one does not assume that G is ergodic on $\Lambda\backslash H$, then we can construct a closed, connected subgroup H_0 of H, such that, after replacing Λ by a conjugate subgroup,
   - H_0 contains G,
   - $(\Lambda \cap H_0)\backslash H_0/(C \cap H_0)$ is a finite cover of M, and
   - G is ergodic on $(\Lambda \cap H_0)\backslash H_0$.

   To see this, write $H = LKR$, where R is the radical of H, K is the maximal compact, semisimple quotient of H, and L is a connected, semisimple subgroup of H, with no compact factors. Let

   - H_0 be the identity component of the closure of $\Lambda L[R, L]$,
   - $\Lambda_0 = \Lambda \cap H_0$, and
   - $C_0 = C \cap H_0$.

   We know that G is ergodic on $\Lambda\backslash H/C$ (because G' is ergodic on P), so, by replacing Λ with a conjugate subgroup, we may assume that ΛGC is dense in H; then $\Lambda H_0 C = H$. Let

   - $\Lambda_1 = \Lambda \cap (H_0C)$,
   - $H_1 = \Lambda_1 H_0$, and
   - $C_1 = C \cap H_1$.

Let us show that the natural map $\Lambda_1\backslash H_1/C_1 \to \Lambda\backslash H/C = M$ is a bijection.

- Because $\Lambda H_0 C = H$, and $H_0 \subset H_1$, we know that the map is surjective.
- Suppose $\lambda h_1 c \in H_1$, with $\lambda \in \Lambda$, $h \in H_1$, and $c \in C$. Then

$$\lambda \in H_1 C H_1 \subset H_0 C,$$

so $\lambda \in \Lambda \cap (H_0 C) = \Lambda_1 \subset H_1$. Therefore $c \in H_1$. So we conclude that the map is injective.

Hence, $\Lambda_1\backslash H_1/C_1$ is diffeomorphic to M.

Now H_1/H_0 is discrete, and contained in the compact group CH_0/H_0, so it must be finite. That is, H_0 has finite index in H_1. Therefore, Λ_0 has finite index in Λ_1, and C_0 has finite index in C_1. Hence, $\Lambda_0\backslash H_0/C_0$ is a finite cover of $\Lambda_1\backslash H_1/C_1 \cong M$. The Mautner phenomenon [6] implies that G is ergodic on $\Lambda_0\backslash H_0$.

COROLLARY 3.13. *In the setting of Corollary* 3.9, *suppose H is semisimple, with no compact factors. Then there exist*

- *a finite-index subgroup* Λ_0 *of* Λ,
- *a connected, compact Lie group* N,
- *a homomorphism* $\sigma\colon \Lambda_0 \to N$, *with dense image,*
- *a quotient* $\overline{C}$ *of* C°, *and*
- *a finite subgroup* F *of the center of* K,

such that $K/F \cong \overline{C} \times N$.

Proof. Let $N = \ker(\rho)^\circ$, so $(H')^\circ = HN$. Because N is a compact, normal subgroup of H', and H has no compact factors, we see that $(H')^\circ$ is isogenous to $H \times N$. By modding out a finite group, let us assume $(H')^\circ = H \times N$.

Let $\Lambda_0 = \Lambda \cap (H')^\circ$, and let $\sigma\colon \Lambda_0 \to N$ be the projection into the second factor of $H \times N$. Because G' has no compact factors, we must have $G' \subset H$. Since G' is ergodic on $\Lambda'\backslash H'$, and, hence, on $\Lambda_0\backslash (H')^\circ$, this implies that $H\Lambda_0$ is dense in $(H')^\circ$, so $\sigma(\Lambda_0)$ is dense in N.

We have $(K')^\circ = \rho^{-1}(C)^\circ = C^\circ \times N$, and $C_0' = C' \cap (K')^\circ$ is a normal subgroup of $(K')^\circ$, such that $(K')^\circ/C_0' \cong K$. Therefore,

$$K \cong \frac{(K')^\circ}{C_0'} \approx \frac{C^\circ N}{C_0' N} \times \frac{N}{N \cap C_0'},$$

where '$\approx$' denotes isogeny, that is, an isomorphism modulo appropriate finite groups. ∎

COROLLARY 3.14. *In the setting of Corollary* 3.9, *suppose that H is semisimple, with no compact factors, that* $\mathbb{R}$-rank $H \geqslant 2$, *and that the lattice* Λ *is irreducible. Then there is a finite list* $K_1, \ldots, K_n$ *of compact groups, depending only on H, such that K is isomorphic to* K_j, *for some j.*

Proof. We apply Corollary 3.13. There are only finitely many connected, compact Lie groups of any given dimension, and $\dim K = (\dim \overline{C}) + (\dim N)$, so it suffices to find bounds on $\dim \overline{C}$ and $\dim N$ that depend only on H. We have $\dim \overline{C} \leqslant \dim H$. Given H, the Margulis Superrigidity Theorem (1.2) implies that there are only finitely many choices for N, up to isomorphism, so $\dim N$ is bounded. ∎

In Corollary 3.14, one may replace the assumption that H is simple with the weaker assumption that

- H is semisimple, and
- the lattice Λ is irreducible.

EXAMPLE 3.15. If $M = \Lambda\backslash H/K$ and $P = \Lambda\backslash H$, then the principal bundle E constructed in the proof of Corollary 3.9 is G'-equivariantly diffeomorphic to $K \times P$, with $(k, p)g = (k, pg)$. So G' is not ergodic on E (unless K is trivial).

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