Ergodic Theory, Groups, and Geometry

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Contents

Preface vii

Acknowledgments ix

Lecture 1. Introduction 1
- 1A. Discrete versions of G 1
- 1B. Nonlinear actions 2
- 1C. Open questions 5
- Comments 5
- References 6

Lecture 2. Actions in Dimension 1 or 2 9
- 2A. Finite actions 9
- 2B. Actions on the circle 10
- 2C. Farb-Shalen method 11
- 2D. Actions on surfaces 12
- Comments 13
- References 14

Lecture 3. Geometric Structures 17
- 3A. Reductions of principal bundles 17
- 3B. Algebraic hull of a G-action on M 18
- 3C. Actions preserving an H-structure 20
- 3D. Groups that act on Lorentz manifolds 21
- Comments 22
- References 22

Lecture 4. Fundamental Groups I 25
- 4A. Engaging conditions 25
- 4B. Consequences 27
- 4C. Examples: Actions with engaging conditions 29
- Comments 30
- References 30

Lecture 5. Gromov Representation 33
- 5A. Gromov's Centralizer Theorem 33
- 5B. Higher-order frames and connections 35
- 5C. Ideas in the proof of Gromov's Centralizer Theorem 37
- Comments 38
- References 38

Lecture 6. Superrigidity and First Applications 41
6A. Cocycle Superrigidity Theorem 41
6B. Application to connection-preserving actions 43
6C. From measurability to smoothness 44
Comments 45
References 45

Lecture 7. Fundamental Groups II (Arithmetic Theory) 47
7A. The fundamental group is a lattice 47
7B. Arithmeticity of representations of the fundamental group 49
Comments 50
References 50

Lecture 8. Locally Homogeneous Spaces 53
8A. Statement of the problem 53
8B. The use of cocycle superrigidity 55
8C. The approach of Margulis and Oh 56
Comments 58
References 58

Lecture 9. Stationary Measures and Projective Quotients 61
9A. Stationary measures 61
9B. Projective quotients 63
9C. Furstenberg entropy 64
Comments 65
References 65

Lecture 10. Orbit Equivalence 67
10A. Definition and basic facts 67
10B. Orbit-equivalence rigidity for higher-rank simple Lie groups 68
10C. Orbit-equivalence rigidity for higher-rank lattice subgroups 68
10D. Orbit-equivalence rigidity for free groups 69
10E. Relationship to quasi-isometries 70
Comments 71
References 72

Appendix: Background Material 75
A1. Lie groups 75
A2. Ergodic Theory 76
A3. Moore Ergodicity Theorem 76
A4. Algebraic Groups 77
A5. Cocycle Superrigidity Theorem 78
A6. Borel Density Theorem 79
A7. Three theorems of G. A. Margulis on lattice subgroups 79
A8. Fixed-point theorems 80
A9. Amenable groups 80
References 81

Name Index 83

Index 85

Preface

The study of group actions on manifolds is the meeting ground of a variety of mathematical areas. In particular, interesting geometric insights can be obtained by applying measure theoretic techniques. These notes provide an introduction to some of the important methods, major developments, and open problems in the subject. They are slightly expanded from lectures of R. J. Z. at a CBMS Conference at the University of Minnesota, Minneapolis, in June, 1998. The main text presents a perspective on the field as it was at that time, and comments after the notes of each lecture provide suggestions for further reading, including references to recent developments, but the content of these notes is by no means exhaustive.

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LECTURE 1

Introduction

When a group G acts on a manifold M, one would like to understand the relation between:

- algebraic properties of the group G,
- the topology of M,
- the G-invariant geometric structures on M, and
- dynamical properties of the action (such as dense orbits, invariant measures, etc.).

If we assume that G is a connected Lie group, then the structure theory (A1.3) tells us there are two main cases to consider:

- *solvable*
  Solvable groups are usually studied by starting with $\mathbb{R}^n$ and proceeding by induction.
- *semisimple*
  There is a classification that provides a list of the semisimple groups ($\mathrm{SL}(n, \mathbb{R})$, $\mathrm{SO}(p, q)$, etc.), so a case-by-case analysis is possible. Alternatively, the groups can sometimes be treated via other categorizations, e.g., by real rank.

The emphasis in these lectures is on the semisimple case.

(1.1) **Assumption.** *In this lecture, G always denotes a connected, noncompact, semisimple Lie group.*

1A. Discrete versions of G

A connected Lie group may have discrete subgroups that approximate it. There are two notions of this that play very important roles in these lectures:

- *lattice*
  This is a discrete subgroup Γ of G such that G/Γ is compact (or, more generally, such that G/Γ has finite volume).
- *arithmetic subgroup*
  Suppose G is a (closed) subgroup of $\mathrm{GL}(n, \mathbb{R})$, and let $\Gamma = G \cap \mathrm{GL}(n, \mathbb{Z})$ be the set of "integer points" of G. If Γ is Zariski dense in G (or, equivalently, if $G \cap \mathrm{GL}(n, \mathbb{Q})$ is dense in G), then we say Γ is an *arithmetic* subgroup of G.

Discrete versions inherit important algebraic properties of G:

(1.2) **Example.** Suppose Γ is a discrete version of G (that is, Γ is either a lattice or an arithmetic subgroup).

1) Because Assumption 1.1 tells us that G is semisimple, we know G has no solvable, normal subgroups (or, more precisely, none that are connected and nontrivial). One can show that Γ also has no solvable, normal subgroups (or, more precisely, none that are infinite, if G has finite center). This follows from the Borel Density Theorem (A6.1), which tells us that Γ is "Zariski dense" in (a large subgroup of) G.
2) If G is simple, then, by definition, G has no normal subgroups (or, more precisely, none that are connected, nontrivial, and proper). If we furthermore assume $\mathbb{R}$-rank$(G) \geq 2$ (i.e., G has a subgroup of dimension ≥ 2 that is diagonalizable over $\mathbb{R}$) and the center of G is finite, then theorems of G. A. Margulis link Γ and G more tightly:
   (a) Γ has no normal subgroups (or, more precisely, none that are infinite and of infinite index), and
   (b) roughly speaking, every lattice in G is arithmetic.

   (See Theorems A7.3 and A7.9 for precise statements.)

(1.3) *Remark.* Both of the conclusions of Example 1.2(2) can fail if we eliminate the assumption that $\mathbb{R}$-rank$(G) \geq 2$. For example, in $\mathrm{SL}(2,\mathbb{R})$, the free group F_2 is a lattice, and some other lattices have homomorphisms onto free groups.

(1.4) **Example** (Linear actions). Let $M = V$, where V is a finite-dimensional vector space (over $\mathbb{R}$ or $\mathbb{C}$). A linear action of G on M (that is, a homomorphism $G \to \mathrm{GL}(V)$) is known as a (finite-dimensional) *representation* of G. These representations are classified by the well-known theory of *highest weights*, so we can think of the linear actions of G as being known.

If $\mathbb{R}$-rank$(G) \geq 2$ (and G is simple), then the linear actions of Γ are closely related to the linear actions of G. Namely, the spirit, but not exactly the statement, of the Margulis Superrigidity Theorem (A7.4) is that if $\varphi\colon \Gamma \to \mathrm{GL}(V)$, then either

- φ extends to a representation of G, or
- the image of φ is contained in a compact subgroup of $\mathrm{GL}(V)$.

In other words, any representation of Γ extends to a representation of G, modulo compact groups. This is another illustration of the close connection between G and its discrete versions.

(1.5) *Remark.* An alternative formulation (A7.7) of the Margulis Superrigidity Theorem says that (in spirit) each representation of Γ either extends to G or extends to G after composing with some field automorphism of $\mathbb{C}$.

1B. Nonlinear actions

These lectures explore some of what is known about the nonlinear actions of G and its discrete versions. (As was mentioned in Example 1.4, the theory of highest weights provides a largely satisfactory theory of the linear actions.) The connections between G and Γ are of particular interest.

We begin the discussion with two basic examples of G-actions.

(1.6) **Example.** G acts on G/Γ, where Γ is any lattice. More generally, if $G \hookrightarrow H$, and Λ is a lattice in H, then G acts by translations on H/Λ.

(1.7) **Example.** The linear action of $\mathrm{SL}(n,\mathbb{R})$ on $\mathbb{R}^n$ factors through to a (nonlinear) action of $\mathrm{SL}(n,\mathbb{R})$ on the projective space $\mathbb{R}P^{n-1}$. More generally, $\mathrm{SL}(n,\mathbb{R})$ acts on Grassmannians, and other flag varieties.

Generalizing further, any semisimple Lie group G has transitive actions on some projective varieties. They are of the form G/Q, where Q is a "parabolic" subgroup of G. (We remark that there are only finitely many of these actions, up to isomorphism, because there are only finitely many parabolic subgroups of G, up to conjugacy.)

(1.8) *Remark.* In the case of $\mathrm{SL}(n,\mathbb{R})$, any parabolic subgroup Q is block upper-triangular:

$$Q = \begin{bmatrix} \boxed{*} & & * \\ & \boxed{*} & \\ 0 & & \boxed{*} \end{bmatrix}$$

A natural question that guides research in the area is:

To what extent is every action on a compact manifold (or perhaps a more general space) built out of the above two basic examples? (1.9)

Unfortunately, there are methods to construct actions that seem to be much less amenable to classification. The two known methods are *Induction* and *Blowing up*; we will briefly describe each of them.

1B(a). Induction. Suppose some subgroup H of G acts on a space Y. Then

1) H acts on $G \times Y$ by
$$h \cdot (g, y) = (gh^{-1}, hy),$$
and

2) G acts on the quotient $X = (G \times Y)/H$ by
$$a \cdot [(g, y)] = [(ag, y)].$$

The action of G on X is said to be *induced* from the action of H on Y.

(1.10) *Remark.*

1) Ignoring the second coordinate yields a G-equivariant map $X \to G/H$, so we see that X is a fiber bundle over G/H with fiber Y. Note that the fiber over $[e] = eH$ is H-invariant, and the action of H on this fiber is isomorphic to the action of H on Y.
2) Conversely, if G acts on X, and there is a G-equivariant map from X to G/H, then X is G-equivariantly isomorphic to an action induced from H. Namely, if we let Y be the fiber of X over $[e]$, then Y is H-invariant, and the map $(g, y) \mapsto g \cdot y$ factors through to a G-equivariant bijection $(G \times Y)/H \to X$.

(1.11) **Example.** Let H be the stabilizer in $\mathrm{SL}(n,\mathbb{R})$ of a point in $\mathbb{R}P^{n-1}$, so

$$H = \left\{ h = \begin{bmatrix} \lambda & * & \cdots & * \\ 0 & & & \\ \vdots & & \boxed{A} & \\ 0 & & & \end{bmatrix} \;\middle|\; \det A = \lambda^{-1} \right\}$$

Then $h \mapsto \log|\lambda|$ is a homomorphism from H onto $\mathbb{R}$, so every $\mathbb{R}$-action yields an action of H; hence, by induction, an action of $\mathrm{SL}(n,\mathbb{R})$.

The upshot is that every vector field on any compact manifold M yields an $\mathrm{SL}(n,\mathbb{R})$-action on a compact manifold M'. Thus, all the complications that arise for $\mathbb{R}$-actions also arise for $\mathrm{SL}(n,\mathbb{R})$-actions.

It is hopeless to classify all $\mathbb{R}$-actions on compact manifolds, so Example 1.11 puts a damper on the hope for a complete classification of actions of simple Lie groups. The following example discourages the belief in a classification theorem even more.

(1.12) **Example.** Let Γ be a lattice in $\mathrm{PSL}(2,\mathbb{R})$, such that Γ has a homomorphism onto a nonabelian free group F. By the same argument as in Example 1.11, we see that every action of F on a compact manifold yields an action of $\mathrm{PSL}(2,\mathbb{R})$ on a manifold (and the manifold is compact if $\mathrm{PSL}(2,\mathbb{R})/\Gamma$ is compact).

In $\mathrm{SL}(n,\mathbb{R})$, let

$$Q = \begin{bmatrix} \boxed{\begin{matrix} * & * \\ * & * \end{matrix}} & \begin{matrix} * & \cdots & * \\ * & \cdots & * \end{matrix} \\ \begin{matrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{matrix} & \boxed{\quad * \quad} \end{bmatrix},$$

where the box in the top left corner is 2×2, so Q has a homomorphism onto $\mathrm{PSL}(2,\mathbb{R})$. Thus,

$$F\text{-action} \to \mathrm{PSL}(2,\mathbb{R})\text{-action} \to Q\text{-action} \to \mathrm{SL}(n,\mathbb{R})\text{-action}.$$

So the free-group problem arises for every $\mathrm{SL}(n,\mathbb{R})$, not just $\mathrm{SL}(2,\mathbb{R})$.

1B(b). Blowing up (Katok-Lewis, Benveniste)**.** Let Γ be a finite-index subgroup of $\mathrm{SL}(n,\mathbb{Z})$, so Γ acts on $\mathbb{T}^n$ by automorphisms, and assume the action has two distinct fixed points x and y.

1) First, *blow up* at these two fixed points. That is, letting $T_p(\mathbb{T}^n)$ be the tangent space to $\mathbb{T}^n$ at p,
   - replace x with $\{\text{rays in } T_x(\mathbb{T}^n)\}$, and
   - replace y with $\{\text{rays in } T_y(\mathbb{T}^n)\}$.

   Thus, x and y have each been replaced with a sphere.
2) Now, glue the two spheres together.

The upshot is that the union of two fixed points can be replaced by a projective space with the usual action of $\mathrm{SL}(n,\mathbb{Z})$ on $\mathbb{R}P^{n-1}$.

(1.13) *Remark.*

1) Blowing up results in a different manifold; the fundamental group is different. (This will be discussed in Example 4.18 below.)
2) The construction can embed the action on a projective space (which is not volume preserving) into a volume-preserving action.
3) The change in the manifold is on a submanifold of positive codimension.

The above construction is due to A. Katok and J. Lewis. It was generalized by J. Benveniste along the following lines.

(1.14) **Example.**

1) Embed G in a larger group H, let $M = H/\Gamma$, for some lattice Λ in H.
2) Assume G is contained in a subgroup L of H, such that L has a closed orbit on H/Γ.

For some pairs of closed L-orbits, we can blow up transversally (that is, take the set of rays in a subspace transverse to the L-orbit), and then glue to make a new action. Varying the gluing results in actions that have nontrivial perturbations.

1C. Open questions

The examples in §1B(a) and §1B(b) suggest there is a limit to what can be done, so we present a few problems that are likely to be approachable. Some will be discussed in later lectures.

1) Are there actions of Γ on low-dimensional manifolds?

   Low-dimensional can either mean low in an absolute sense, as in dimension ≤ 3, or it can mean low relative to Γ, which is often taken to be less than

   $$(\text{lowest dimension of a representation of } G) - 1$$

   For example, if $\Gamma = \mathrm{SL}(n, \mathbb{Z})$, then "low-dimensional" means either ≤ 3 or $< n - 1$.
2) When are actions locally rigid?

   Not all actions are locally rigid. For example, if the action is induced from a vector field, then it may be possible to perturb the vector field. (Note that perturbations may be non-linear actions, even if the original action is linear.)
3) Suppose G preserves a geometric structure on M, defined by a structure group H. Then what is the relation between G and H?

   Sometimes, assuming that G preserves a suitable geometric structure eliminates the examples constructed above.
4) Does every action have either:
   - an invariant geometric structure of rigid type (at least, on an open set), or
   - an equivariant quotient G-manifold with such a structure?
5) If G acts on M, what can be said about the fundamental group $\pi_1(M)$? What about other aspects of the topology of M?
6) What are the consequences of assuming there is a G-invariant volume form on M?

Approaches to these questions must bear in mind the constructions of §1B(a) and §1B(b) that provide counterexamples to many naive conjectures.

Comments

Proofs of the fundamental theorems of G. A. Margulis mentioned in Examples 1.2 and 1.4 can be found in [**9, 10, 11**].

The blowing-up construction of §1B(b) is due to A. Katok and J. Lewis [**7**]. Example 1.14 appeared in the unpublished Ph. D. thesis of E. J. Benveniste [**1**].

See [**4**] for details of a stronger result. A few additional examples created by a somewhat different method of gluing appear in [**6**, §4].

Some literature on the open questions in Section 1C:

1) Actions on manifolds of low dimension ≤ 2 are discussed in Lecture 2 (and the comments at the end).
2) See [**3**] for a recent survey of the many results on local rigidity of group actions.
3) Actions with an invariant geometric structure are discussed in Lectures 3, 5 and 6.
4) The examples of Katok-Lewis and Benveniste do not have a rigid geometric structure that is invariant [**2**]. On the other hand, the notion of "almost-rigid structure" is introduced in [**2**], and all known volume-preserving, smooth actions of higher-rank simple Lie groups have an invariant structure of that type.
5) Results on the fundamental group of M are discussed in Lectures 4, 5 and 7.
6) Many of the results discussed in these lectures apply only to actions that are volume preserving (or, at least, have an invariant probability measure). Only Lecture 9 is specifically devoted to actions that are *not* volume preserving.

See the survey of D. Fisher [**5**] (and other papers in the same volume) and the ICM talk of F. Labourie [**8**] for a different view of several of the topics that will be discussed in these lectures.

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LECTURE 2

Actions in Dimension 1 or 2

Suppose Γ is a lattice in a connected, noncompact, simple Lie group G. When M has small dimension, one can sometimes prove that every action of Γ on M is nearly trivial, in the following sense:

(2.1) **Definition.** An action of a group Γ is *finite* if the kernel of the action is a finite-index subgroup of Γ.

2A. Finite actions

(2.2) *Remark.*

1) It is easy to see that an action of Γ is finite if and only if it factors through an action of a finite group. More explicitly, every finite action of Γ on a space M can be obtained from the following construction:
   - let F be a finite group that acts on M, so we have a homomorphism $\varphi_1 \colon F \to \mathrm{Homeo}(M)$, and
   - let $\varphi_2 \colon \Gamma \to F$ be any homomorphism.

   Then the composition $\varphi = \varphi_1 \circ \varphi_2 \colon \Gamma \to \mathrm{Homeo}(M)$ defines a finite action of Γ on M.
2) It is known that an action of a finitely generated group on a connected manifold is finite if and only if every orbit of the action is finite. (One direction of this statement is obvious, but the other is not.)

It is useful to know that, in certain situations, the existence of a single finite orbit implies that every orbit is finite:

(2.3) **Proposition.** *Suppose that every n-dimensional linear representation of every finite-index subgroup of the lattice Γ is finite. If a smooth action of Γ on a connected n-manifold M has a finite orbit, then the action is finite.*

The proof utilizes the following fundamental result:

(2.4) **Lemma** (Reeb-Thurston Stability Theorem). *Suppose the group Λ is finitely generated and acts by C^1 diffeomorphisms on a connected manifold M, with a fixed point p. If*

- *Λ acts trivially on the tangent space $T_p(M)$, and*
- *there is no nontrivial homomorphism from Λ to $\mathbb{R}$,*

then the action is trivial (i.e., every point of M is fixed by every element of Λ).

Idea of proof. To simplify the proof, let us assume that the action is real analytic (i.e., C^ω). By choosing coordinates, we may assume $M = \mathbb{R}^n$, and $p = 0$. To avoid cumbersome notation, let us assume $n = 1$.

Now, for each $\lambda \in \Lambda$, there is some $a_\lambda \in \mathbb{R}$, such that

$$\lambda(x) = x + a_\lambda x^2 + o(x^2).$$

(Note that the leading term is x, because, by assumption, Λ acts as the identity on the tangent space $T_0\mathbb{R}$.) For $\lambda, \gamma \in \Lambda$, it is easy to verify that

$$a_{\lambda\gamma} = a_\lambda + a_\gamma.$$

That is, the map $\lambda \mapsto a_\lambda$ is a homomorphism. By assumption, this implies $a_\lambda = 0$ for all $\lambda \in \Lambda$.

By induction on k, the argument of the preceding paragraph implies

$$\lambda(x) = x + o(x^k), \quad \text{for every } k.$$

Since the action is real analytic, we conclude that $\lambda(x) = x$, so the action is trivial. □

Proof of Proposition 2.3. Because Γ has a finite orbit, we may choose a finite-index subgroup Γ_1 of Γ that has a fixed point p. Then Γ_1 acts on the tangent space T_pM. This is an n-dimensional representation of Γ_1, so, by assumption, we may choose a finite-index subgroup Γ_2 of Γ_1 that acts trivially on T_pM. Also, since every n-dimensional representation of Γ_2 is finite, it is clear that Γ_2 has no nontrivial homomorphisms into $\mathbb{R}$. So the Reeb-Thurston Theorem implies that Γ_2 fixes every point in M; i.e., Γ_2 is in the kernel of the action. □

For lattices of higher real rank, the Margulis Superrigidity Theorem (A7.4) implies that every 1-dimensional or 2-dimensional linear representation is finite. Thus, we have the following corollary:

(2.5) **Corollary.** *Suppose Γ is a lattice in a simple Lie group G with $\mathbb{R}$-rank$(G) \geq 2$ (and the center of G is finite). If a smooth Γ-action on a circle or a surface has a finite orbit, then the action is finite.*

In many cases, the Margulis Normal Subgroup Theorem (A7.3) tells us that every normal subgroup of Γ is either finite or of finite index. In this situation, any action with an infinite kernel must be finite, so we have the following useful observation:

(2.6) **Lemma.** *Suppose Γ is a lattice in a simple Lie group G, with $\mathbb{R}$-rank$(G) \geq 2$ (and the center of G is finite). If Γ acts on M, and the action of some infinite subgroup of Γ is finite, then the action of Γ is finite.*

2B. Actions on the circle

The conclusion of Corollary 2.5 can sometimes be proved without assuming that there is a finite orbit. Let us start by stating the results for actions on a circle.

(2.7) **Theorem** (Witte). *Suppose Γ is any finite-index subgroup of $\mathrm{SL}(n, \mathbb{Z})$, with $n \geq 3$. Then every C^0 Γ-action on a circle is finite.*

More generally, the conclusion is true whenever Γ is an arithmetic lattice with $\mathbb{Q}$-rank$(\Gamma) \geq 2$. (Unfortunately, however, because of the assumption on $\mathbb{Q}$-rank, this result does not apply to any cocompact lattice in any simple Lie group.) It should be possible to replace the assumption on $\mathbb{Q}$-rank with an assumption on $\mathbb{R}$-rank:

(2.8) **Conjecture.** *Let Γ be a lattice in a simple Lie group G with $\mathbb{R}\text{-rank}(G) \geq 2$. Then every C^0 Γ-action on a circle is finite.*

(2.9) **Theorem** (Ghys, Burger-Monod). *Conjecture 2.8 is true for C^1 actions.*

Witte's proof is algebraic, and uses the linear order of the real line. Ghys's proof studies the action of Γ on the space of probability measures on the circle, and shows it has a fixed point. Burger and Monod prove a vanishing theorem for bounded cohomology (namely, $H^1_{\mathrm{bdd}}(\Gamma;\mathbb{R})$ injects into $H^1(\Gamma;\mathbb{R})$), and obtain the conjecture as a corollary (in cases where $H^1(\Gamma;\mathbb{R}) = 0$).

We will prove only the following less-general result, because it is all that we need to obtain an interesting result for actions on surfaces.

(2.10) **Theorem** (Farb-Shalen). *Conjecture 2.8 is true for C^ω actions of a special class of lattices.*

2C. Farb-Shalen method

Corollary 2.5 assumes the existence of a finite Γ-orbit. B. Farb and P. Shalen showed that, in certain cases, it suffices to know that a single element γ of Γ has a finite orbit. An element with a finite orbit can sometimes be found by topological methods, such as the Lefschetz Fixed-Point Formula.

(2.11) **Definition** (Farb-Shalen). Let Γ be a lattice in a simple Lie group G with $\mathbb{R}\text{-rank}(G) \geq 2$. We call Γ *big* if there exist

- an element γ of Γ of infinite order, and
- a subgroup Λ of the centralizer $C_\Gamma(\gamma)$, such that Λ is isomorphic to a lattice in some simple Lie group of real rank at least two.

(2.12) *Remark.*

1) Every finite-index subgroup of $\mathrm{SL}(n,\mathbb{Z})$ is big if $n \geq 5$.
2) There exist big cocompact lattices in some simple Lie groups.
3) There are non-big lattices in simple Lie groups of arbitrarily large real rank.

We can now state and prove Farb-Shalen's theorem on the circle.

(2.13) **Theorem** (Farb-Shalen). *If Γ is big, then every C^ω action of Γ on the circle is finite.*

Proof. Because Γ is big, we may choose γ and Λ as described in the definition. The classic Denjoy's Theorem implies that there are two cases to consider.

> **Denjoy's Theorem.** *If f is a C^∞ diffeomorphism of S^1, then either*
>
> 1) *f has a finite orbit, or*
> 2) *f is C^0-conjugate to an irrational rotation.*

Case 1. Assume γ has a finite orbit. There is some $k > 0$, such that γ^k has a fixed point. Because the action is C^ω, the fixed points of γ^k are isolated, so the fixed-point set of γ^k is finite. Also, because Λ centralizes γ^k, this fixed-point set is Λ-invariant. Thus, Λ has a finite orbit. Then Corollary 2.5 tells us that the Λ-action is finite, so the entire Γ-action is finite (see Lemma 2.6).

Case 2. Assume γ is C^0-conjugate to an irrational rotation. For simplicity, let us assume, by replacing γ with a conjugate, that γ is an irrational rotation. The powers of γ are dense in the group of all rotations, so Λ centralizes all the rotations of S^1. This implies that Λ acts by rotations, so the commutator subgroup $[\Lambda, \Lambda]$ acts trivially. Because $[\Lambda, \Lambda]$ is infinite, we again conclude that the Γ-action is finite (see Lemma 2.6). □

(2.14) *Remark.* In the proof of Theorem 2.13, real analyticity was used only in Case 1, to show that the fixed-point set of γ^k is finite. Without this assumption, the fixed-point set could be any closed subset of S^1 (such as a Cantor set).

2D. Actions on surfaces

(2.15) **Conjecture.** *Let Γ be a lattice in a simple Lie group G with $\mathbb{R}$-rank$(G) \geq 2$. Then every area-preserving C^0 Γ-action on any (compact, connected) surface is finite.*

(2.16) *Remark.* The assumption that the action is area-preserving cannot be omitted, because the natural action of $\mathrm{SL}(3,\mathbb{Z})$ on $\mathbb{R}^3$ factors through to a faithful action on the sphere S^2.

(2.17) **Definition.** Let Γ be a lattice in a simple Lie group G with $\mathbb{R}$-rank$(G) \geq 2$. We call Γ *very big* if there exist

- an element γ of Γ of infinite order, and
- a subgroup Λ of the centralizer $C_\Gamma(\gamma)$, such that Λ is isomorphic to a **big** lattice in some simple Lie group of real rank at least two.

(2.18) *Remark.*

1) Every finite-index subgroup of $\mathrm{SL}(n,\mathbb{Z})$ is very big if $n \geq 7$.
2) There exist very big cocompact lattices in some simple Lie groups.

(2.19) **Theorem** (Farb-Shalen). *Assume Γ is very big. Then a C^ω action of Γ on a surface S is finite if either*

- $\chi(S) \neq 0$ (*non-vanishing Euler characteristic*), *or*
- *the action is area-preserving.*

(2.20) *Remark.* The first theorem of this type was obtained by É. Ghys, who showed that if Γ belongs to a certain class of *non*cocompact lattices (e.g., if Γ is a finite-index subgroup of $\mathrm{SL}(n,\mathbb{Z})$, for $n \geq 4$), then every C^ω action of Γ on the 2-sphere is finite. This is a corollary of his theorem that every nilpotent group of C^ω diffeomorphisms of S^2 is metabelian.

The Farb-Shalen proof for actions on surfaces is based on the same idea as their proof for actions on the circle. Given $\gamma \in \Gamma$, consider the fixed-point set M^γ. This is an analytic set, i.e., it is closed and defined locally by zeros of analytic functions. Analytic sets need not be manifolds (i.e., there may be singularities), but we have the following basic lemma.

(2.21) **Lemma** (Farb-Shalen). *If $V \subset M$ is a nonempty analytic set, then $\exists W \subset V$, such that*

1) *W is a compact analytic submanifold of V, and*
2) *W is canonical: if $f\colon M \to M$ is any C^ω diffeomorphism, such that $f(V) \subset V$, then $f(W) \subset W$.*

Proof of the Farb-Shalen Theorem for surfaces. Choose γ and Λ as in the definition of very big. If $\chi(S) \neq 0$, it follows easily from the Lefschetz Fixed-Point Formula (A8.1) that γ^k has a fixed point, for some $k > 0$. If $\chi(S) = 0$, then it is not so easy to get a fixed point, but, by using the entire group Γ, and the assumption that the action is area-preserving, it is possible to apply a fixed-point criterion of J. Franks (see Theorem A8.2).

Thus, we may assume that γ^k has a fixed point, for some $k > 0$. Then, by applying Lemma 2.21 to the fixed-point set of γ^k, we see that Λ acts on either a circle or a finite set. Because actions of big lattices on a circle (or on a finite set) are finite, we conclude that Λ has a finite orbit, so the Λ-action on M is finite (see Corollary 2.5). Hence, Lemma 2.6 implies that the Γ-action is finite. □

(2.22) *Remark.* Using the Ghys-Burger-Monod Theorem (2.9) yields the conclusion of the Farb-Shalen Theorem for actions of big lattices on surfaces, not only actions of very big lattices.

Comments

See [**20**, Thm. 1] for a proof of Remark 2.2(2).

The Reeb-Thurston Stability Theorem (2.4) was first proved in [**19**]. See [**17**] or [**18**] for a nice proof.

See [**10**] or [**11**] for an introduction to actions of lattices on the circle. Theorem 2.7 was proved in [**22**]. Theorem 2.9 was proved by É. Ghys [**9**] and (in most cases) M. Burger and N. Monod [**1, 2**]. For C^2 actions, a generalization to all groups with Kazhdan's property (T) was proved by A. Navas [**12**].

Conjecture 2.8 remains open. Indeed, there is not a single known example of a (torsion-free) cocompact lattice Γ, such that no finite-index subgroup of Γ has a faithful C^0 action on the circle. On the other hand, Theorem 2.7 was generalized to many *non*cocompact lattices in [**13**].

The Farb-Shalen method, including a proof of Theorem 2.19, was introduced in [**4**]. It is now known that if Γ is a *non*cocompact lattice in a simple Lie group G, such that Γ contains a torsion-free, nonabelian, nilpotent subgroup, and $\mathbb{R}\text{-rank}(G) \geq 2$, then every smooth, area-preserving Γ-action on any surface is finite. This was proved by L. Polterovich [**15**] for surfaces of genus ≥ 1 (without assuming the existence of a nonabelian nilpotent subgroup), and the general case was obtained by J. Franks and M. Handel [**7**].

The results in Remark 2.20 appear in [**8**]. It was later shown by J. C. Rebelo [**16**] that these results for C^ω actions on S^2 are also true for homotopically trivial C^ω actions on $\mathbb{T}^2$. This fills in the missing case of Theorem 2.19 (real-analytic actions on $\mathbb{T}^2$ that are not area-preserving), for a class of *non*cocompact lattices that includes all finite-index subgroups of $\mathrm{SL}(4, \mathbb{Z})$.

The fixed-point theorem (A8.2) of J. Franks appears in [**6**].

B. Farb and H. Masur [**3**] showed that if Γ is a lattice in a simple Lie group G, with $\mathbb{R}$-rank$(G) \geq 2$, then any action of Γ on a closed, oriented surface is homotopically trivial on a finite-index subgroup Γ' (i.e., the homeomorphism induced by any element of Γ' is isotopic to the identity).

Moving beyond surfaces, Farb and Shalen [**5**] proved that if Γ is a finite-index subgroup of $\mathrm{SL}(8,\mathbb{Z})$ (or, more generally, if Γ is a lattice of $\mathbb{Q}$-rank at least 7 in a simple Lie group), then every real-analytic, volume-preserving action of Γ on any compact 4-manifold of nonzero Euler characteristic is finite.

Assume $k < n$ (and $n \geq 3$). S. Weinberger [**21**] showed that $\mathrm{SL}(n,\mathbb{Z})$ has no nontrivial C^∞ action on $\mathbb{T}^k$, and K. Parwani [**14**] showed that any action of $\mathrm{SL}(n+1,\mathbb{Z})$ on a mod 2 homology k-sphere is finite. These proofs make use of elements of finite order in the acting group.

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LECTURE 3

Geometric Structures

(3.1) **Assumption.** *G is a noncompact, simple Lie group acting smoothly* (*and nontrivially*) *on an n-manifold M.*

In this lecture, we discuss just a few of the ideas that arise when the action of G preserves a pseudo-Riemannian metric, or some other geometric structure on M.

3A. Reductions of principal bundles

Recall that the frame bundle PM is a principal $\mathrm{GL}(n,\mathbb{R})$-bundle over M.

(3.2) **Definition.** Suppose $P \to M$ is a principal H-bundle, and L is a subgroup of H. A *reduction of P to L* is any of the following three equivalent objects:

1) a principal L-subbundle Q of P,
2) a section $s\colon M \to P/L$, or
3) an H-equivariant map $\varphi\colon P \to H/L$.

Proof of equivalence. ($1 \Rightarrow 2$) The quotient Q/L has a single point in each fiber of P/L, so it defines a section of P/L.

($2 \Rightarrow 3$) Let $\pi\colon P \to M$ be the bundle map. For each $x \in P$, the points xL and $s\big(\pi(x)\big)$ are in the same fiber of P/L, so there exists $\eta_x \in H$, such that $s\big(\pi(x)\big) = x\,\eta_x L$. Although η_x is (usually) not unique, the coset $\eta_x L$ is well defined, so η defines a map $\overline{\eta}\colon P \to H/L$. For $h \in H$, it is clear that $\eta_{xh} = h^{-1}\eta_x$, so η is H-equivariant. (The inverse on h reflects the fact that we have a right action on P, but a left action on H/L.)

($3 \Rightarrow 1$) Let $Q = \{\, x \in P \mid \varphi(x) = eL \,\}$. □

(3.3) *Remark.*

1) A reduction is always assumed to be measurable. It may or may not be required to have some additional regularity: C^0, C^∞, etc.
2) When we have chosen a measure μ on M, we ignore sets of measure 0: the section $M \to P/L$ need only be defined a.e. on M.

(3.4) **Definition.** Let H be a subgroup of $\mathrm{GL}(n,\mathbb{R})$. An *$H$-structure on M* is a reduction of PM to H.

(3.5) *Remark.* In applications, H is usually an algebraic group; i.e., it is Zariski closed in $\mathrm{GL}(n,\mathbb{R})$.

(3.6) **Example.** If $H = \mathrm{O}(p,q)$, then an H-structure on M is equivalent to the choice of a pseudo-Riemannian metric on M of signature (p,q).

PROOF. ($\Leftarrow$) We are given a pseudo-Riemannian metric $\langle\ |\ \rangle$ of signature (p,q). For each $m \in M$, let Q_m be the set of all frames $\{v_1, v_2, \ldots, v_{p+q}\}$ of $T_m M$, such that the corresponding Cartan matrix of $\langle\ |\ \rangle_m$ is

$$\begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{bmatrix}.$$

I.e., we have

$$\langle v_i \mid v_j \rangle = \begin{cases} 1 & \text{if } i = j \le p \\ -1 & \text{if } p < i = j \\ 0 & \text{if } i \ne j. \end{cases}$$

Then Q is an $\mathrm{O}(p,q)$-subbundle of PM. □

Any action of G on M by diffeomorphisms lifts to an action on the tangent bundle TM. It also lifts to an action on the frame bundle PM. (The derivative of a diffeomorphism takes a frame on M to another frame on M.) Thus, we can speak of *G-invariant H-structures* on M.

(3.7) **Question.** *If ω is an H-structure on M, then what is* $\mathrm{Aut}(M,\omega)$ *(i.e., the set of diffeomorphisms preserving ω)?*

Especially, we would like to know whether $\mathrm{Aut}(M,\omega)$ contains a copy of G. In other words, is there a G-action on M that preserves the H-structure ω?

3B. Algebraic hull of a G-action on M

(3.8) **Theorem.** *Let G act on a principal H-bundle $P \to M$, with H an algebraic group, and assume μ is an ergodic G-invariant measure on M.*

Then there is an algebraic subgroup L of H, such that

1) *there is a G-invariant reduction to L,*
2) *there is no G-invariant reduction to any proper algebraic subgroup of L, and*
3) *whenever there is a G-invariant reduction to an algebraic subgroup J of H, then J contains a conjugate of L.*

(3.9) *Remark.* It is clear from (1) and (3) that the subgroup L is unique up to conjugacy.

To prepare for the proof, we recall a few basic facts about tame actions.

(3.10) **Definition.**

1) A topological space Y is *tame* if it is second countable and T_0. (Recall that "T_0" means $\forall x, y \in Y$, $\exists$ open subset U of Y, such that U contains either x or y, but not both.)
2) An H-action on Y is *tame* if the orbit space Y/H is a tame topological space.

(3.11) **Example.**

1) Any proper action on a locally compact, second countable, Hausdorff space is tame.
2) Any algebraic action on a variety is tame (see Corollary A4.4).

(3.12) **Lemma.**

1) *Each atomic measure on a tame space Y is supported on a single point.* (*"Atomic" means that Y cannot be decomposed into two sets of positive measure.*)
2) *If $f\colon M \to Y$ is a G-invariant measurable function, G is ergodic on M, and Y is tame, then f is constant (a.e.).*

Proof of Theorem 3.8. From the descending chain condition on Zariski-closed subgroups of H, it is clear that there is a minimal algebraic subgroup L of H, such that there is a G-invariant reduction to L. This subgroup L obviously satisfies (1) and (2).

We now prove (3). Suppose

$$\phi_1\colon P \to H/L \text{ and } \phi_2\colon P \to H/J$$

are both H-equivariant and G-invariant. Let

$$\phi = (\phi_1, \phi_2)\colon P \to H/L \times H/J$$

and

$$\overline{\phi}\colon M \to (H/L \times H/J)/H.$$

Because H acts algebraically on the variety $H/L \times H/J$, we know that the orbit space $(H/L \times H/J)/H$ is tame. We also know that G is ergodic on M, so we conclude that $\overline{\phi}$ is constant (a.e.). This means the image of ϕ is (a.e.) contained in a single H-orbit on $H/L \times H/J$.

The stabilizer of a point in $H/L \times H/J$ is (conjugate to) $L \cap (hJh^{-1})$, for some $h \in H$, so we can identify the H-orbit with $H/\big(L \cap (hJh^{-1})\big)$. Thus, after discarding a set of measure 0 from M, we may think of ϕ as a map $P \to H/\big(L \cap (hJh^{-1})\big)$ that is G-invariant and H-equivariant. Thus, ϕ is a G-invariant reduction of P to $L \cap (hJh^{-1})$. Then the minimality of L implies $L \cap (hJh^{-1}) = L$, so L is contained in a conjugate of J. Therefore, a conjugate of L is contained in J. □

(3.13) **Definition.** The subgroup L of the theorem is the *algebraic hull* of the G-action on P. It is well defined up to conjugacy.

(3.14) *Remark.* The above is the *measurable* algebraic hull. For actions with a dense orbit, the same argument shows there is a C^r-hull with the same properties, but the reductions are defined only on open, dense, G-invariant sets.

(3.15) **Example** (Computation of an algebraic hull). Suppose G acts on M, and ρ is a homomorphism from G to H. Define an action of G on the trivial principal bundle $P = M \times H$ by

$$g \cdot (m, h) = \big(gm, \rho(g)h\big).$$

We will show that the algebraic hull of this action is the Zariski closure $\overline{\overline{\rho(G)}}$ of $\rho(G)$ in H.

Note that, for any subgroup L of H, a section of the trivial bundle P/L can be thought of as being simply a map $M \to H/L$. Furthermore, the section is G-invariant iff the corresponding map is G-*equi*variant.

($\subset$) Since $\rho(G)$ fixes the point $[e]$ in $H/\overline{\overline{\rho(G)}}$, the constant map $m \mapsto [e]$ is G-equivariant, so it yields a G-invariant section of $P/\overline{\overline{\rho(G)}}$. Therefore, the algebraic hull is contained in $\overline{\overline{\rho(G)}}$.

($\supset$) Let $L \subset \overline{\overline{\rho(G)}}$ be the algebraic hull of the action. By definition, this implies there is a G-invariant reduction to L, so there is a G-equivariant map $f\colon M \to \overline{\overline{\rho(G)}}/L$. Then $f_*\mu$ is an ergodic G-invariant measure on $\overline{\overline{\rho(G)}}/L$, so the Borel Density Theorem (A6.1) implies that $\rho(G)$ fixes a.e. point in $\overline{\overline{\rho(G)}}/L$. Hence, $\rho(G)$ is contained in some conjugate of L. Since the algebraic hull L is Zariski closed, we conclude that it contains $\overline{\overline{\rho(G)}}$.

3C. Actions preserving an H-structure

The following result places a strong restriction on the geometric structures that can be preserved by a G-action. (Recall that $n = \dim M$, so, by definition, if there is an H-structure on M, then $H \subset \mathrm{GL}(n,\mathbb{R})$.)

(3.16) **Theorem.** *If G preserves an H-structure and a finite measure on M, then there is* (*locally*) *an embedding $\rho\colon G \to H$.*

Moreover, the composite homomorphism

$$G \xrightarrow{\rho} H \hookrightarrow \mathrm{GL}(n,\mathbb{R})$$

defines a representation of G that contains the adjoint representation of G.

The proof employs a lemma that is of independent interest:

(3.17) **Lemma.** *If G acts on M with finite invariant measure and has no fixed points* (*except on a set of measure* 0)*, then, for almost every $m \in M$, the stabilizer G_m of m is discrete.*

Proof. For each $m \in M$, let $\mathfrak{s}(m)$ be the Lie algebra of G_m. Then $\mathfrak{s}(m)$ is a linear subspace of the Lie algebra $\mathfrak{g}$, so $\mathfrak{s}$ is a map from G to a Grassmannian variety $\mathcal{V}$. Because stabilizers are conjugate, it is easy to see that $\mathfrak{s}\colon M \to \mathcal{V}$ is G-equivariant, where the action of G on $\mathcal{V}$ is via the adjoint representation. Then $\mathfrak{s}_*\mu$ is an $\operatorname{Ad} G$-invariant probability measure on $\mathcal{V}$, so the Borel Density Theorem (A6.1) implies that the image of $\mathfrak{s}$ is (a.e.) contained in the fixed-point set of the adjoint representation. Since G is simple, the only $\operatorname{Ad} G$-invariant subspaces of $\mathfrak{g}$ are $\{0\}$ and $\mathfrak{g}$. But G has no fixed points, so the Lie algebra of a stabilizer cannot be $\mathfrak{g}$. We conclude that the Lie algebra of almost every stabilizer is $\{0\}$. □

Proof of Theorem 3.16. From the lemma, we know that the stabilizer of almost every point of M is discrete, so, for (almost) every point m in M, the differential of the map $G \to Gm$ yields an embedding of $\mathfrak{g}$ in $T_m M$. Thus, TM contains, as a subbundle, the vector bundle $M \times \mathfrak{g}$ with G-action defined by

$$g \cdot (m, v) = \big(gm, (\operatorname{Ad} g)v\big).$$

From Example 3.15, we know that the algebraic hull of the action on the principal bundle associated to this subbundle is the Zariski closure of $\operatorname{Ad} G$. The algebraic hull on PM, the principal bundle associated to TM, must have the algebraic hull of this subbundle as a subquotient, so it contains $\operatorname{Ad} G$. □

3D. Groups that act on Lorentz manifolds

The adjoint representation of G can almost never be embedded in a representation with an invariant Lorentz metric, so Theorem 3.16 has the following consequence.

(3.18) **Corollary** (Zimmer). *If there is a G-invariant Lorentz metric on M, and M is compact, then G is locally isomorphic to* $\mathrm{SL}(2,\mathbb{R})$.

More generally, here is a list (up to local isomorphism) of all the Lie groups that can act on a compact Lorentz manifold:

(3.19) **Theorem** (Adams-Stuck, Zeghib). *Let H be a connected Lie group that acts faithfully on a compact manifold M, preserving a Lorentz metric. Then H is locally isomorphic to $L \times C$, where C is compact, and L is either*

i) $\mathrm{SL}(2,\mathbb{R})$,

ii) *the $ax + b$ group (i.e.,* $\mathbb{R}^\times \ltimes \mathbb{R}$),

iii) *a Heisenberg group (of dimension $2k+1$, for some k),*

iv) *a semidirect product* $\mathbb{R} \ltimes (\text{Heisenberg})$, *or*

v) *the trivial group* $\{e\}$.

(3.20) *Remark.*

1) In (iv), the possible actions of $\mathbb{R}$ on the Heisenberg normal subgroup can be described explicitly, but we omit the details.

2) Conversely, if H is locally isomorphic to $L \times C$, as in Theorem 3.19 (with the additional restriction indicated in (1)), then some connected Lie group locally isomorphic to H has a faithful action on a compact manifold M, preserving a Lorentz metric.

3) H is locally isomorphic to the full isometry group of a compact Lorentz manifold iff H is as described in Theorem 3.19, but with option (ii) deleted.

One can say a lot about G, even without assuming that M is compact:

(3.21) **Theorem** (Kowalsky). *If there is a G-invariant Lorentz metric on M, and G has finite center, then either:*

i) *the G-action is not proper, and G is locally isomorphic to* $\mathrm{SO}(1,n)$ *or* $\mathrm{SO}(2,n)$, *or*

ii) *the G-action is proper, and the cohomology group $H^k(M;\mathbb{R})$ vanishes whenever $k \geq n-p$, where $p = \dim G/K$ is the dimension of the symmetric space associated to G.*

(3.22) **Example.** Let $M = (\text{compact}) \times \mathbb{R}$. Then $\mathrm{SL}(3,\mathbb{R})$ cannot act on M preserving a Lorentz structure (because $H^{n-1}(M;\mathbb{R}) \neq 0$).

Comments

See [**15**] for a more thorough introduction to the use of ergodic theory in the study of actions preserving a geometric structure. This is a very active field; Lecture 6 and the references there present some additional developments.

An analogue of Theorem 3.16 for actions that preserve a Cartan geometry appears in [**4**]

The algebraic hull first appeared in [**13**].

In the situation of Lemma 3.17, if we assume that $\mathbb{R}\text{-rank}\, G \geq 2$, and that the action is ergodic, but not transitive (a.e.), then a.e. stabilizer is trivial, not merely discrete [**10**].

See [**1**] for a detailed discussion of actions on Lorentz manifolds. Corollary 3.18 appears in [**14**]. Theorem 3.19 appears in [**2, 11**] (together with its converse). Remark 3.20(3) appears in [**3, 12**]. Theorem 3.21 appears in [**8**].

The study of actions on Lorentz manifolds remains active. For example, see [**6**] for a discussion of the Lorentz manifolds that admit nonproper actions by semisimple groups of isometries.

Lorentz manifolds are pseudo-Riemannian manifolds of signature $(1, n-1)$. Theorem 3.16 can be used to show that if G is a simple group acting by isometries on a more general compact pseudo-Riemannian manifold M of signature (p, q), then $\mathbb{R}\text{-rank}\, G \leq \min(p, q)$. A result of R. Quiroga-Barranco [**9**] describes the structure of M in the extreme case where $\mathbb{R}\text{-rank}\, G = \min(p, q)$. There has also been progress [**5, 7**] in understanding actions on pseudo-Riemannian manifolds by simple groups of conformal maps.

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LECTURE 4

Fundamental Groups I

This lecture is a first look at the relation between G and the fundamental group of a manifold on which it acts. We will see more on this subject in Lectures 5 and 7.

(4.1) **Assumption.** *In this lecture, we are given:*

1) *a connected, noncompact, semisimple Lie group G,*
2) *an action of G on a compact manifold M, such that the fundamental group $\pi_1(M)$ is infinite, and*
3) *an ergodic, G-invariant probability measure μ on M.*

We assume, for simplicity, that G is simply connected, so G acts on the universal cover $\widetilde{M}$.

4A. Engaging conditions

(4.2) *Remark.*

1) The universal cover $\widetilde{M}$ is a principal bundle over M, with fiber $\pi_1(M)$.
2) If Λ is any subgroup of $\pi_1(M)$, then $\widetilde{M}/\Lambda$ is another cover of M.

Choosing a fundamental domain yields a measurable section $M \to \widetilde{M}/\Lambda$. (The section is continuous on a dense, open set if the fundamental domain is reasonably nice.)

(4.3) **Definition.** By a *reduction* of $\widetilde{M}$ to Λ, we mean a measurable section $M \to \widetilde{M}/\Lambda$.

(4.4) **Example.** Let $M = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. The universal cover $\widetilde{M}$ is $\mathbb{R}^2$, and a section $s\colon \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2$ is given by

$$s\big([(x,y)]\big) = \big(\{x\},\{y\}\big),$$

where $\{\cdot\}$ denotes the fractional part.

(4.5) **Example.** Let H be a connected, simply connected group that acts on the unit square $[0,1]^2$, fixing each point on the boundary.

- By gluing opposite sides of the square, we obtain an action of H on $\mathbb{T}^2$.
- The lift to an action on $\widetilde{\mathbb{T}^2} = \mathbb{R}^2$ is a tiling by copies of the original action: for $n \in \mathbb{Z}^2$ and $x \in [0,1]^2$, we have $g \cdot (x+n) = (g \cdot x) + n$.

Thus, there is an H-equivariant reduction of $\widetilde{\mathbb{T}^2}$ to the trivial subgroup $\{e\}$.

An "engaging condition" is a property that prohibits this type of reduction.

(4.6) **Definition.** The action of G on M is:

1) *totally engaging* if there is no G-equivariant reduction of $\widetilde{M}$ to any *proper* subgroup of $\pi_1(M)$;
2) *engaging* if there is no G-equivariant reduction of $\widetilde{M}$ to any proper subgroup of *finite index* in $\pi_1(M)$;
3) *topologically engaging* if there is no G-equivariant reduction of $\widetilde{M}$ to any *finite* subgroup of $\pi_1(M)$.

(4.7) *Remark.* The definition of *engaging* is vacuous unless $\pi_1(M)$ has at least one proper subgroup of finite index. Under the stronger hypothesis that $\pi_1(M)$ has infinitely many subgroups of finite index, it is not difficult to see that

$$\text{totally engaging} \implies \text{engaging} \implies \text{topologically engaging.}$$

(4.8) *Remark.*

1) Suppose there is a G-equivariant reduction of $\widetilde{M}$ to $\Lambda \subset \pi_1(M)$.
   (a) If the action of G on M is totally engaging, then $\Lambda = \pi_1(M)$.
   (b) If the action of G on M is engaging, then Λ is profinitely dense in $\pi_1(M)$, i.e., Λ surjects into every finite quotient of $\pi_1(M)$. (This implies that, although Λ may have infinite index in $\pi_1(M)$, it is a "very big" subgroup of $\pi_1(M)$.)
2) For many actions, there does not exist a smallest subgroup Λ of $\pi_1(M)$, such that there is a G-equivariant reduction of $\widetilde{M}$ to Λ. This is because the set of discrete subgroups usually does not have the descending chain condition. Hence, we cannot define the "discrete hull" of an action on a general manifold M.
3) The phrase "G-equivariant" in Definition 4.6 is an abuse of terminology — we really mean "G-equivariant *almost everywhere*" (with respect to the G-invariant measure μ).

Usually an action is ergodic for a good reason that is insensitive to finite covers. For example, Anosov flows are ergodic, and the Anosov condition, being local, is not affected by passing to a finite cover. So the following proposition shows that engaging is a natural assumption from a dynamical point of view.

(4.9) **Proposition.** *The G-action on M is engaging iff G is ergodic on every finite cover of M.*

Proof. ($\Leftarrow$) Suppose the action is not engaging. Then there is a G-equivariant section $s\colon M \to \widetilde{M}/\Lambda$, for some finite-index, proper subgroup Λ of $\pi_1(M)$. Let $E = s(M) \subset \widetilde{M}/\Lambda$. Then:

- E is a G-invariant set, because s is G-equivariant, and
- E is neither null nor conull, because

$$\mu(E) = \mu(M) = \frac{\mu\big(\widetilde{M}/\Lambda\big)}{|\pi_1(M) : \Lambda|}$$

is neither 0 nor $\mu\big(\widetilde{M}/\Lambda\big)$.

Therefore, the G-action on $\widetilde{M}/\Lambda$ is not ergodic.

($\Rightarrow$) Suppose G is not ergodic on the finite cover $\widehat{M}$. This means there is a G-invariant subset E of $\widehat{M}$ that is neither null nor conull. For $m \in M$, let

$$k(m) = \#\big(p^{-1}(m) \cap E\big), \text{ where } p\colon \widehat{M} \to M \text{ is the covering map.}$$

Since E is a G-invariant set, it is clear that k is a G-invariant function. Since G is ergodic on M, this implies that k is constant (a.e.). Hence, $\mu(E)$ is an integer multiple of $\mu(M)$, so we may assume $\mu(E)$ is minimal among the (non-null) G-invariant subsets of $\widetilde{M}$.

Assuming, without loss of generality, that $\widehat{M}$ is a regular cover, i.e., that $M = \widehat{M}/\Lambda$, for some finite group Λ of deck transformations, we may let

$$\Lambda' = \{\, \lambda \in \Lambda \mid E = \lambda E \text{ (a.e.)} \,\}.$$

It is immediate that Λ' is a subgroup of Λ. Furthermore, it follows from the minimality of $\mu(E)$ that $\mu(E \cap \lambda E) = 0$, for every $\lambda \in \Lambda \smallsetminus \Lambda'$. This implies that the image of E in $\widehat{M}/\Lambda'$ is a (G-invariant) fundamental domain for the covering map $\widehat{M}/\Lambda' \to M$ (a.e.). So the action is not engaging. □

The following is an easy way to show that an action is topologically engaging. In fact, this condition is usually taken to be the *definition* of "topologically engaging" (which makes it possible to obtain some of the results without any mention of measure theory).

(4.10) **Proposition.** *Assume that the support of μ has nonempty interior. If G (or even a single element of infinite order in G) acts properly on a dense, open subset $\mathcal{O}$ of $\widetilde{M}$, then the action is topologically engaging.*

PROOF. Let $\Lambda = \pi_1(M)$ and assume, for simplicity, that $\mathcal{O}$ is Λ-invariant. The action of G on $\mathcal{O}$ is tame, so it is also tame on $\mathcal{O}' = \mathcal{O}/\Lambda_0$, for any finite subgroup Λ_0 of Λ.

Suppose s is a G-equivariant reduction of $\widetilde{M}$ to Λ_0. The image of $\mathcal{O}'$ in M is not null (because the support of μ has nonempty interior), so ergodicity implies that it is conull. Therefore, $s_*\mu$ is an ergodic measure on $\mathcal{O}'$, so, by tameness, it puts all of its mass in a single G-orbit. Letting G_x be the stabilizer of a point in this orbit, we see that

- G/G_x has finite volume (because $s_*\mu$ is a G-invariant probability measure), and
- G_x is compact (because G acts properly on $\mathcal{O}'$). □

This implies that G has finite volume, which contradicts our standing assumption that G is noncompact.

4B. Consequences

After recalling the definition of amenability, we prove a theorem that illustrates the use of engaging conditions.

(4.11) **Definition.** Let Λ be a discrete group.

1) The (left) *regular representation* of Λ is the action of Λ on the Hilbert space $\ell^2(\Lambda)$ by (left) translation:

$$(\lambda \cdot \varphi)(x) = \varphi(\lambda^{-1}x).$$

This is an action by unitary operators.

2) Λ is *amenable* if the regular representation of Λ has "almost invariant vectors." That is, for every finite subset F of Λ, and every $\epsilon > 0$, there is a unit vector v in $\ell^2(\Lambda)$, such that $\forall \lambda \in F$, we have $\|\lambda v - v\| < \epsilon$.

(See Remark A9.3 for alternative definitions of amenability.)

(4.12) **Example.** $\mathbb{Z}$ is amenable, because the normalized characteristic function of a long interval moves very little under translations of a bounded size.

(4.13) **Theorem.** *Suppose the action of G on M satisfies any of the engaging conditions. If $\mathbb{R}$-rank$(G) \geq 2$, and G is simple, then $\pi_1(M)$ cannot be amenable.*

Proof. For convenience, let $\Lambda = \pi_1(M)$. Then $\widetilde{M}$ is a bundle over M with fiber Λ. The group G acts by bundle automorphisms, so, when an element of G maps one fiber to another, the only twisting of the fiber is via multiplication (on the left, say) by an element of Λ.

Suppose Λ is amenable. We can view $L^2(\widetilde{M})$ as the sections of a bundle over M with fiber $\ell^2(\Lambda)$. The conclusion of the preceding paragraph implies that the map from fiber to fiber in $L^2(\widetilde{M})$ is via translation by an element of Λ (the regular representation). Then, since

- Λ is amenable, and
- constant functions are in $L^2(M)$ (because M has finite measure),

we see, without too much difficulty, by choosing a constant section $M \to \ell^2(\Lambda)$ whose value is an almost invariant vector in $\ell^2(\Lambda)$, that $L^2(\widetilde{M})$ has almost invariant vectors.

On the other hand, because $\mathbb{R}$-rank$(G) \geq 2$, we know that G has Kazhdan's property (T), which means, by definition, that if a unitary representation of G has almost invariant vectors, then the representation has invariant vectors. Therefore, there is a G-invariant unit vector in $L^2(\widetilde{M})$. Equivalently, there is a G-invariant set E of (nonzero) finite measure in $\widetilde{M}$.

Now, the proof of Proposition 4.9($\Rightarrow$) implies there is a G-invariant reduction to a finite subgroup of $\pi_1(M)$. This contradicts all of the engaging hypotheses. $\square$

The above proof uses an assumption of amenability to obtain almost invariant vectors in the regular representation. For some nonamenable groups, there are other representations that have almost invariant vectors.

(4.14) **Proposition.** *Let $H = \mathrm{SL}(2,\mathbb{R})$, or $\mathrm{SO}(1,n)$ or $\mathrm{SU}(1,n)$. Then there is a unitary representation $(\rho, \mathcal{H})$ of H, such that*

1) *ρ has almost invariant vectors, and*
2) *H acts properly on $\mathcal{H} - \{0\}$.*

Sketch of proof. The existence of representations satisfying Statement (1) is a consequence of the fact that H does not have Kazhdan's property (T). Statement (2) follows from the decay of matrix coefficients for unitary representations of semisimple groups, proved by R. Howe, C. C. Moore, and T. Sherman. $\square$

This leads to the following result:

(4.15) **Theorem.** *Suppose the action of G on M satisfies any of the engaging conditions. If $\mathbb{R}\text{-rank}(G) \geq 2$, then $\pi_1(M)$ cannot be an infinite discrete subgroup of $\mathrm{SO}(1,n)$ or $\mathrm{SU}(1,n)$. In particular, $\pi_1(M)$ cannot be a surface group, a free group, or the fundamental group of a real hyperbolic manifold.*

Proof. Let $\Lambda = \pi_1(M)$, and suppose Λ is an infinite discrete subgroup of $H = \mathrm{SO}(1,n)$ or $\mathrm{SU}(1,n)$. By restricting the representation $(\rho, \mathcal{H})$ of H provided by Proposition 4.14, we obtain a representation of Λ with the same properties. Let

$$L^2_\Lambda(\widetilde{M}; \mathcal{H}) = \{\, \Lambda\text{-equivariant } L^2 \text{ functions } \widetilde{M} \to \mathcal{H} \,\}.$$

(We can think of this as the L^2 sections of a bundle over M with fiber $\mathcal{H}$.)

We have a representation of G by translations on $L^2_\Lambda(\widetilde{M}; \mathcal{H})$, and the argument in the first three paragraphs of the proof of Theorem 4.13 implies there is a G-invariant vector in $L^2_\Lambda(\widetilde{M}; \mathcal{H})$.

Since the action of Λ on $\mathcal{H} \smallsetminus \{0\}$ is proper, the same tameness arguments used in Lecture 3 provide a G-invariant section of a bundle whose fiber is a single Λ-orbit, and we can identify this orbit with Λ/Λ_0, where Λ_0 is the stabilizer of some nonzero vector $f \in \ell^2(\Lambda)$. This yields a G-equivariant reduction of $\widetilde{M}$ to Λ_0. Because the stabilizer of every nonzero vector in $\mathcal{H}$ is finite (because the action is proper), we know that Λ_0 is finite. This contradicts the engaging condition. □

4C. Examples: Actions with engaging conditions

(4.16) **Example.** Let $M = H/\Lambda$, where G embeds in the simple Lie group H as a closed subgroup, and Λ is a lattice in H.

i) The action is topologically engaging. Because G acts properly on H, it acts properly on the universal cover of H, which is the same as the universal cover $\widetilde{M}$ of M. Therefore, it is obvious from Proposition 4.10 that the action is topologically engaging.

ii) The action is engaging. Any finite cover of M is a coset space H/Λ', for some finite-index subgroup Λ' of Λ. Because Λ' is a lattice in H, the Moore Ergodicity Theorem (A3.3) implies that G is ergodic on H/Λ'. Therefore, the action is engaging, by Proposition 4.9.

iii) The action is totally engaging. A cover of M is a coset space H/Λ', for some subgroup Λ' of Λ. (Note that Λ' can be *any* subgroup of Λ, perhaps of infinite index, so it might not be a lattice in H.) Suppose $s\colon H/\Lambda \to H/\Lambda'$ is a G-equivariant section.

Because Λ is a lattice in H, there is an H-invariant probability measure μ on H/Λ. The Moore Ergodicity Theorem implies that μ is G-ergodic, so $s_*\mu$ is an ergodic G-invariant probability measure on H/Λ'.

Ratner's Theorem (A2.3) implies that there is a closed, connected subgroup L of H, such that $s_*\mu$ is supported on a single L-orbit. Because $s_*\mu$ must project to μ, which gives measure 0 to any immersed submanifold of lower dimension, we must have $L = H$.

However, Ratner's Theorem also tells us that $s_*\mu$ is L-invariant, so, because $L = H$, we conclude that $s_*\mu$ is H-invariant. Thus, H/Λ' has an H-invariant probability measure, namely $s_*\mu$, so Λ' is a lattice in H. Thus, Λ' has finite index in Λ, so H/Λ' is a finite cover of $H/\Lambda = M$.

Since the action is engaging, and $s\colon M \to H/\Lambda'$ is a G-equivariant section, this implies $H/\Lambda' = M$, so $\Lambda' = \Lambda$.

M. Gromov provided another important class of topologically engaging examples. We will see the proof in Lecture 5.

(4.17) **Theorem** (Gromov). *If G preserves a connection and a volume form on M, and everything is real analytic, then the action is topologically engaging.*

(4.18) **Example.** The Katok-Lewis and Benveniste examples constructed by blowing up are **not** engaging. For example, there is a double-cover of the Katok-Lewis example that, after throwing out a set of measure 0, consists of two copies of the original action. (The set of measure 0 consists of the spheres along which the two copies are glued together.) Therefore, the double cover is not ergodic, so the action is not engaging.

Here is a construction of such a double cover. The Katok-Lewis example M is constructed from a certain space X (obtained by blowing up two fixed points) that contains two distinguished spheres A and B, by gluing A to B. Let X' be a copy of X, with distinguished spheres A' and B'. Now glue A to B' and glue A' to B, resulting in a space Y that is a double cover of M. (Identifying X with X' collapses Y to M.) If we remove A and B from Y, then the gluing is also removed, so we obtain X and X' with their distinguished spheres removed.

On the other hand, it seems that all known examples of Katok-Lewis-Benveniste type are topologically engaging, because there are covers on which the action of G is proper. In particular, if the original manifold (before the blowing up) has a cover that contains a dense, open set on which G acts properly, then this yields a cover of the Katok-Lewis-Benveniste example with a dense, open subset on which G acts properly.

The above example illustrates the difference between engaging and topological engaging:

- For an action to be topologically engaging, there only needs to be *some* subgroup of the fundamental group that yields a proper action, and the rest of the fundamental group is irrelevant.
- In contrast, the condition for engaging sees *all* of the fundamental group (if $\pi_1(M)$ is residually finite).

Comments

The notions of *engaging* and *topologically engaging* were introduced in [**5**]. See Lecture 7 for additional references and a discussion of results from [**3**], where the definition of *totally engaging* first appeared. A notion of *engaging* for actions of discrete groups was introduced in [**2**].

Amenability and Kazhdan's property (T) appear in many texts. Thorough discussions of the two topics can be found in [**4**] and [**1**], respectively.

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LECTURE 5

Gromov Representation

Suppose we have a volume-preserving action of G on a compact manifold M. As in Lecture 4, we are interested in understanding the fundamental group of M. If we assume that the action is real-analytic and has an invariant connection, a theorem of M. Gromov provides a very interesting finite-dimensional representation of $\pi_1(M)$. The image of this representation contains a copy of G in its Zariski closure, and the representation can be used to show that the action of G on M is topologically engaging.

5A. Gromov's Centralizer Theorem

(5.1) **Notation.** Let $\widetilde{M}$ be the universal cover of M and let $\Lambda = \pi_1(M)$.

(5.2) *Remark.* One can hope to learn something about Λ by constructing an interesting finite-dimensional representation; that is, by finding an interesting homomorphism from Λ to $\mathrm{GL}(n, \mathbb{R})$. Specifically, in this lecture, we would like to find a representation ρ of Λ, such that the Zariski closure of $\rho(\Lambda)$ is large. This imposes a significant restriction on the possibilities for the fundamental group Λ. For example, a superrigid lattice in a group H does not have any homomorphism whose image has a large Zariski closure, because the Zariski closure of the image is basically H (but, maybe with a compact group added on).

Given $\rho \in \mathrm{Hom}(\Lambda, H)$, where H is any algebraic group, one can form the associated principal bundle P_ρ over M with fiber H:

$$P_\rho = (\widetilde{M} \times H)/\Lambda.$$

Note that G acts on P_ρ (via its action on the factor $\widetilde{M}$). The following observation shows that if the algebraic hull of this action is "large," then the Zariski closure of $\rho(\Lambda)$ must also be "large."

(5.3) **Lemma.** *For any $\rho \in \mathrm{Hom}(\Lambda, H)$, the algebraic hull of the G-action on P_ρ is contained in the Zariski closure $\overline{\overline{\rho(\Lambda)}}$ of $\rho(\Lambda)$.*

Proof. Because $\big(\widetilde{M} \times \overline{\overline{\rho(\Lambda)}}\big)/\Lambda$ is a G-invariant subbundle of P_ρ, which means P_ρ has a reduction to $\overline{\overline{\rho(\Lambda)}}$, we know that the algebraic hull is contained in $\overline{\overline{\rho(\Lambda)}}$. □

The action of Λ by deck transformations on $\mathrm{Vect}(\widetilde{M})$, the space of vector fields on $\widetilde{M}$, is a noteworthy representation of Λ. However, this representation is infinite dimensional, so the above discussion does not apply to it. For connection-preserving actions that are real analytic, the following important theorem provides an interesting finite-dimensional subrepresentation of $\mathrm{Vect}(\widetilde{M})$, and we will see that the algebraic hull of the associated principal bundle is quite large (namely, it contains a group that is locally isomorphic to G).

(5.4) **Theorem** (Gromov's Centralizer Theorem). *Assume*

- *G acts on a compact manifold M with fundamental group Λ,*
- *G is simple and noncompact (but not necessarily of higher rank), and*
- *M has a connection and a volume form, each of which is C^ω and G-invariant.*

Let V be the set of all vector fields X in $\operatorname{Vect}(\widetilde{M})$*, such that*

a) *X centralizes $\mathfrak{g}$ and*

b) *X preserves the connection.*

Then

1) *V is Λ-invariant,*

2) $\dim V < \infty$,

3) *V centralizes $\mathfrak{g}$, and*

4) $\forall x \in \widetilde{M}$, $\operatorname{ev}_x(V) \supset T\,Gx$, *where* $\operatorname{ev}_x \colon \operatorname{Vect}(\widetilde{M}) \to T_x\widetilde{M}$ *is the evaluation map* $X \mapsto X(x)$.

(5.5) *Remark.* Condition (a) means that the vector field X is G-invariant. Condition (b) means that the connection is invariant under the local one-parameter group generated by X.

(5.6) **Example.** Suppose $M = G/\Lambda$, so $\widetilde{M} = G$. Then $\mathfrak{g}$ is the space of right-invariant vector fields on G, so V is the space of left-invariant vector fields on G.

(5.7) *Remark* (Comments on the proof of Theorem 5.4).

- Conclusion (1) is obvious because Λ centralizes G and the connection is Λ-invariant (being lifted from a connection on M).
- Conclusion (2) is the "rigidity of connections." It follows from the fact that any connection defines a Riemannian metric on the frame bundle, and the fact that the isometry group of a Riemannian manifold is finite dimensional.
- Conclusion (3) is immediate from the definition of V.
- Conclusion (4) is not at all obvious; it is not even clear that there is a single nonzero vector field that commutes with G.

The proof of the theorem is difficult, because of Conclusion (4); a sketch will be given in Section 5C. The crucial point is the computation of the algebraic hull by using the Borel Density Theorem not just in the tangent bundle, but also in higher-order frame bundles.

Here are some consequences:

(5.8) **Corollary** (Gromov). *In the setting of Theorem 5.4, the fundamental group Λ has a a finite-dimensional representation ρ, such that $\overline{\overline{\rho(\Lambda)}}$ contains a subgroup locally isomorphic to G.*

PROOF. From (1) and (2), we know that the action of Λ on V defines a finite-dimensional representation ρ.

Let P be the principal bundle associated to the vector bundle $(\widetilde{M} \times V)/\Lambda$. The evaluation map $\mathrm{ev}\colon \widetilde{M} \times V \to T\widetilde{M}$ is Λ-equivariant, so, by passing to the quotient, we obtain a vector-bundle map

$$\overline{\mathrm{ev}}\colon (\widetilde{M} \times V)/\Lambda \to TM.$$

From (3), it is clear that $\overline{\mathrm{ev}}$ is G-equivariant. Then, by using (4), we see from the proof of Theorem 3.16 that the algebraic hull of the action on P must contain a group locally isomorphic to G. Since Lemma 5.3 tells us that this algebraic hull is contained in $\overline{\overline{\rho(\Lambda)}}$, this completes the proof. □

(5.9) **Example.** Suppose $G = \mathrm{SL}(4,\mathbb{R})$ in the setting of Theorem 5.4. Then $\pi_1(M)$ cannot be a lattice in $\mathrm{SL}(3,\mathbb{R})$.

PROOF. Let Λ be a lattice in $\mathrm{SL}(3,\mathbb{R})$, and let ρ be any representation of Λ. The Margulis Superrigidity Theorem (A7.4) implies that $\overline{\overline{\rho(\Lambda)}}$ has a finite-index subgroup that is locally isomorphic to $\mathrm{SL}(3,\mathbb{R}) \times (\text{compact})$ — it certainly does not contain $\mathrm{SL}(4,\mathbb{R})$. □

We can now sketch the proof of Theorem 4.17:

(5.10) **Corollary** (Gromov). *In the setting of Theorem 5.4, the action of G on M is topologically engaging.*

IDEA OF PROOF. By passing to a dense, open subset of M, let us assume that the stabilizer of every point in M is discrete (cf. Lemma 3.17). Let

- P be the principal bundle associated to the vector bundle $(\widetilde{M} \times V)/\Lambda$, and
- Q be the principal bundle associated to $T(G\text{-orbits})$.

Then there is an embedding of $\widetilde{M}$ in P, and Q is a quotient of a subbundle of P.

The bundle Q contains $M \times G$, where G acts diagonally: $a \cdot (m,g) = (am, ag)$. Therefore, G is obviously proper on Q. So it is also proper on P. This implies that G is proper on $\widetilde{M}$. (Actually, because of technicalities that have been ignored here, one can conclude only that G is proper on a dense, open subset of $\widetilde{M}$.) □

5B. Higher-order frames and connections

As background to the proof of Gromov's Centralizer Theorem (5.4), we now discuss how to think of a connection in terms of H-structures. As mentioned in Example 3.6, a pseudo-Riemannian metric is more-or-less obviously equivalent to an $\mathrm{O}(p,q)$-structure; a connection is an H-structure for a higher-order frame bundle.

(5.11) **Notation.**

- Let $\phi, \psi\colon \mathbb{R}^n \to M$ be defined in a neighborhood of 0. We write $\phi \sim_k \psi$ if the derivatives up to order k agree in some neighborhood of 0. An equivalence class is a *k-jet*. (In particular, a 0-jet is the same as a "germ" of a function.)
- $J^k(\mathbb{R}^n, 0; M)$ is the set of all k-jets.
- $P^{(k)}(M)$ is the subset of $J^k(\mathbb{R}^n, 0; M)$ consisting of the k-jets of local diffeomorphisms. It is called the *kth-order frame bundle*.

(5.12) *Remark.* $P^{(k)}(M)$ is a bundle over M.

- The map $P^{(k)}(M) \to M$ evaluates a k-jet at 0.
- The fiber over m is the set of all k-jets of local diffeomorphisms $(\mathbb{R}^n, 0) \leftrightarrow (M, m)$. Each point in the fiber is a "kth-order frame at m."

Note that $P^{(k)}(M)$ is a natural bundle, by which we mean that every diffeomorphism of M yields a diffeomorphism of $P^{(k)}(M)$.

(5.13) **Example.** Consider $P^{(1)}(M)$. For $[f] \in P^{(1)}(M)$, the derivative $df_0 \colon \mathbb{R}^n \to TM_m$ is an isomorphism, so $P^{(1)}(M)_m$ is the set of all linear isomorphisms from $\mathbb{R}^n$ to TM_m. Thus, $P^{(1)}(M)$ is simply the frame bundle $P(M)$.

(5.14) **Definition.** We let $\mathrm{GL}^{(k)}$ be the set of all k-jets of local diffeomorphisms $(\mathbb{R}^n, 0) \leftrightarrow (\mathbb{R}^n, 0)$.

(5.15) *Remark.*

- $\mathrm{GL}^{(k)}$ acts on $P^{(k)}(M)$ (on the right), so it is easy to see that $P^{(k)}(M)$ is a principal $\mathrm{GL}^{(k)}$-bundle.
- $\mathrm{GL}^{(1)} = \{\text{invertible 1st derivatives}\} = \mathrm{GL}(n, \mathbb{R})$.
- The map $f \mapsto (df_0, D^2 f_0)$ yields a bijection from $\mathrm{GL}^{(2)}$ onto $\mathrm{GL}(n, \mathbb{R}) \times S$, where $S = S^2(\mathbb{R}^n; \mathbb{R}^n)$ is the space of symmetric $\mathbb{R}^n$-valued 2-tensors on $\mathbb{R}^n$. This shows that $\mathrm{GL}^{(2)}$ is a semidirect product $\mathrm{GL}(n, \mathbb{R}) \ltimes S$, for some action of $\mathrm{GL}(n, \mathbb{R})$ on the abelian group S.

(5.16) **Exercise.** Show $\mathrm{GL}^{(k)} \cong \mathrm{GL}(n, \mathbb{R}) \ltimes U$, for some unipotent group U.

A reduction of $P^{(k)}(M)$ to a subgroup of $\mathrm{GL}^{(k)}$ is a "kth-order geometric structure."

(5.17) **Proposition.** *A torsion-free connection on M is equivalent to a reduction of the $\mathrm{GL}^{(2)}$-bundle $P^{(2)}(M)$ to $\mathrm{GL}(n, \mathbb{R})$.*

Proof. ($\Leftarrow$) Suppose we have a reduction of $P^{(2)}(M)$ to $\mathrm{GL}(n, \mathbb{R})$. This means that, for each $m \in M$, we have a 2-jet $[\phi] \colon (\mathbb{R}^n, 0) \to (M, m)$; the 2-jet is not well-defined, but it is well-defined up to composition with an element of $\mathrm{GL}(n, \mathbb{R})$, because the reduction is to $\mathrm{GL}(n, \mathbb{R})$.

Given $X, Y \in \mathrm{Vect}(M)$ near m, we want to define the covariant derivative $D_X Y$. Let

$$D_X Y = \phi_* \big(\nabla_{\phi_*^{-1} X} (\phi_*^{-1} Y) \big),$$

where ∇ is the ordinary covariant derivative on $\mathbb{R}^n$. The value of $D_X Y$ does not change if ϕ is replaced with ϕA, for any $A \in \mathrm{GL}(n, \mathbb{R})$, so D is well defined. □

(5.18) **Proposition.** *Suppose G acts on M, and let $L^{(2)} \subset \mathrm{GL}^{(2)}$ be the algebraic hull of the action of G on $P^{(2)}(M)$. If $L^{(2)}$ is reductive, then there is a measurable G-invariant connection.*

Proof. Exercise 5.16 implies that any reductive subgroup of $\mathrm{GL}^{(2)}$ is contained in (a conjugate of) $\mathrm{GL}(n, \mathbb{R})$. □

(5.19) *Remark.*

1) If we use the C^r-algebraic hull, instead of the measurable hull, in Proposition 5.18, then there is a connection that is C^r on an open dense subset of M.
2) Assume $\mathbb{R}\text{-rank}(G) \geq 2$ (and G is simple). Then the measurable algebraic hull of the G-action on any principal bundle is reductive with compact center. Therefore, Proposition 5.18 implies that every action of G has a measurable invariant connection.

5C. Ideas in the proof of Gromov's Centralizer Theorem

Assume G acts on M, preserving a connection ω. To prove Theorem 5.4, we need a method to generate vector fields on $\widetilde{M}$ that preserve ω.

(5.20) **Notation.**

- Let $P^{(k)}$ be the kth-order frame bundle over M (with fiber $\mathrm{GL}^{(k)}$).
- The kth-order tangent bundle $T^{(k)}M$ is an associated vector bundle for $P^{(k)}$; the fiber $T^{(k)}M_x$ is the space of $(k-1)$-jets of vector fields at x.
- Let $L^{(k)}$ be the algebraic hull of the G-action on $P^{(k)}$, the "kth-order algebraic hull," so $L^{(k)} \subset \mathrm{GL}^{(k)}$.

In Theorem 3.16, we showed that G embeds in $L = L^{(1)}$, and the homomorphism $G \to L$ contains Ad. We use the same argument, but in kth order instead of first order.

(5.21) **Proposition.** *We have $L^{(k)} \supset G$, and, in the concrete representation of $L^{(k)}$ on $T^{(k)}M$, the representation of G contains* Ad.

Proof. For each $x \in M$, we have $\mathfrak{g}_x^{(k)} \subset T^{(k)}M_x$, where $\mathfrak{g}_x^{(k)}$ is the space of $(k-1)$-jets of elements of the Lie algebra $\mathfrak{g}$, considered as a subspace of $\mathrm{Vect}(M)$. Furthermore, the G-action on $\mathfrak{g}^{(k)}$ is the adjoint action. Therefore, the the desired conclusion follows from Example 3.15, as in the proof of Theorem 3.16. □

Because G also preserves the connection ω, we can say more about $L^{(k)}$. This allows us to show, at each point of M, that there are enough automorphisms up to order k to contain all of $\mathrm{Ad}(G)$.

(5.22) **Notation.** Let $\mathrm{Aut}^{(k)}(M, \omega; x)$ be the set of k-jets of local diffeomorphisms at x that preserve ω up to order k.

(5.23) **Proposition.** *We have* $\mathrm{Aut}^{(k-2)}(M, \omega; x) \supset \mathrm{Ad}_{\mathfrak{g}^{(k)}} G$.

Proof. The connection ω is a reduction of a $\mathrm{GL}^{(2)}$-bundle to $\mathrm{GL}^{(1)}$. Since $\mathrm{GL}^{(2)}$ acts on connections at a point, the group $\mathrm{GL}^{(k)}$ acts on $(k-2)$-jets of connections at a point.

Because G preserves the connection ω, it preserves a section of the bundle whose fiber at each point is the space of $(k-2)$-jets of connections. Roughly speaking, this means there is a reduction of the associated principal bundle to $\mathrm{Aut}^{(k-2)}(M, \omega; x)$. Therefore

$$\mathrm{GL}^{(k)} \supset \mathrm{Aut}^{(k-2)}(M, \omega; x) \supset L^{(k)}.$$

Now the desired conclusion is obtained by combining this with Proposition 5.21, which tells us that $L^{(k)} \supset \operatorname{Ad}_{\mathfrak{g}^{(k)}} G$. □

Three steps remain:

Step 1. Pass from $\operatorname{Aut}^{(k-2)}(M, \omega; x)$ *to* $\operatorname{Aut}^{(\mathrm{loc})}(M, \omega; x)$. That is, take the infinitesimal automorphisms and turn them into local automorphisms. This is achieved by using the Frobenius Theorem ("rigid properties of connections"); passing from infinitesimal automorphisms to local automorphisms is a well-developed topic in Differential Geometry.

Step 2. Pass from local automorphisms on $\widetilde{M}$ *to global automorphisms.* This is based on analytic continuation. For vector fields preserving connections, this type of result appears in Nomizu.

(5.24) *Remark.* We now know that there exist many new vector fields on $\widetilde{M}$ preserving ω.

- By "many," we mean that the vector fields normalize $\mathfrak{g}$, and the representation on $\mathfrak{g}$ contains $\operatorname{Ad} G$.
- By "new," we mean that these vector fields leave x fixed (or, we should say that they vanish at x), so no two of them differ by an element of $\mathfrak{g}$.

Step 3. Go from normalizing to centralizing. This step uses the simplicity of G: an element of the normalizer differs from the centralizer only by an element of G.

Comments

The results of M. Gromov discussed in this lecture originally appeared in [**6**]. (In particular, the key inclusion (4) of Theorem 5.4 is [**6**, 5.2.A$_2$], and Corollary 5.10 is [**6**, 6.1.B$_1$].) The survey [**10**] includes a discussion of this material.

See [**3, 4**] and [**2**] for detailed proofs of Gromov's Centralizer Theorem (5.4) and Corollary 5.10, respectively. Step 1 of the proof of Theorem 5.4 is the gist of a result known as Gromov's Open-Dense Theorem (see [**4**, Thm. 4.3(2)]). It first appeared in [**6**, Thm. 1.6.F], and an exposition of the proof is given in [**1**].

It is conjectured [**2**, §5] that connection-preserving actions are not only topologically engaging, but also "geometrically engaging," which is a stronger property.

Remark 5.19(2) was proved in [**9**], by using the Cocycle Superrigidity Theorem (6.8).

The classical work of K. Nomizu [**8**] discusses how to extend vector fields, as required in Step 2 of the proof of Theorem 5.4.

See [**7**, Thm. 1.1] for a generalization of Corollary 5.8 that does not assume the volume form on M is G-invariant. See [**5**] for an analogue of Corollary 5.8 that requires only a Γ-action, rather than a G-action, where Γ is a higher-rank lattice.

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LECTURE 6

Superrigidity and First Applications

Suppose a simple Lie group G acts by bundle automorphisms on a principal H-bundle P over M. The Cocycle Superrigidity Theorem is an explicit description of this action, in the case where H is an algebraic group and $\mathbb{R}$-rank$(G) \geq 2$.

6A. Cocycle Superrigidity Theorem

Any (measurable) section $s\colon M \to P$ yields a trivialization of P (i.e., an isomorphism of P with the trivial H-bundle $M \times H$). The isomorphism from $M \times H$ to P is given by $(m, h) \mapsto s(m) \cdot h$.

(6.1) **Question.** *Can we choose s so that the resulting action of G on $M \times H$ is very simple?*

Given $m \in M$ and $g \in G$, the two points $g \cdot s(m)$ and $s(gm)$ are both in the fiber over gm, so they differ by an element of H; there is an element $\eta(g, m) \in H$ with

$$g \cdot s(m) = s(gm) \cdot \eta(g, m). \tag{6.2}$$

Under the trivialization $P \cong M \times H$ defined by s, the G-action on $M \times H$ is given by

$$g \cdot (m, h) = \big(gm, \eta(g, m)h \big).$$

Hence, to address Question 6.1, we would like to find a section s that makes the formula for $\eta(g, m)$ as simple as possible.

(6.3) **Definition.** Let ρ be a homomorphism from G to H. We call the section s *totally ρ-simple* if

$$g \cdot s(m) = s(gm) \cdot \rho(g) \quad \text{for every } g \in G \text{ and } m \in M.$$

(6.4) *Remark.*

1) If s is totally ρ-simple, then the corresponding G-action on $M \times H$ is given by

$$g \cdot (m, h) = \big(gm, \rho(g)h \big).$$

So a totally ρ-simple section "diagonalizes" the G-action.

2) Equation (6.2) implies that the function $(g, m) \mapsto \eta(g, m)$ satisfies the *cocycle identity*

$$\eta(g_1 g_2, m) = \eta(g_1, g_2 m)\, \eta(g_2, m).$$

Thus, a reader familiar with cocycles of dynamical systems may note that the search for a totally ρ-simple section is exactly the search for a homomorphism cohomologous to η (see Definition A5.1).

(6.5) **Example.** Suppose G acts locally freely on M (i.e., the stabilizer of each point in M is discrete). If we let

- $T(G\text{-orbits})$ be the subbundle of TM consisting of vectors that are tangent to a G-orbit, and
- P be the frame bundle of $T(G\text{-orbits})$,

then there is a smooth totally Ad-simple section of P. To see this, note that, for each $x \in M$, the differential of the map $G \to Gx$ is an isomorphism of the Lie algebra $\mathfrak{g}$ with $T(G\text{-orbits})_x$, because G acts locally freely. Therefore, letting $E = M \times \mathfrak{g}$, we have an isomorphism from E to $T(G\text{-orbits})$. For $g \in G$, $v \in \mathfrak{g}$, and $m \in M$, we have

$$\begin{aligned}\frac{d}{dt}\Big(g \exp(tv)\, m\Big)\Big|_{t=0} &= \frac{d}{dt}\Big(\big(g \exp(tv)\, g^{-1}\big) \cdot gm\Big)\Big|_{t=0} \\ &= \frac{d}{dt}\Big(\exp\big(t\,(\operatorname{Ad} g)v\big) \cdot gm\Big)\Big|_{t=0},\end{aligned}$$

so the G-action on E is given by

$$g \cdot (m, v) = \big(gm, (\operatorname{Ad} g)v\big).$$

We then clearly have a similar statement for the associated principal bundle P.

(6.6) **Example.** Suppose Γ is a lattice in G and $\sigma\colon \Gamma \to H$ is a homomorphism. Let

$$P = P_\sigma = (G \times H)/\Gamma,$$

a principal H-bundle over G/Γ. It is not difficult to show that

$$\begin{aligned}&\text{there is a homomorphism } \rho\colon G \to H \text{ and a totally } \rho\text{-simple section of } P \\ &\qquad \iff \sigma \text{ extends to a homomorphism } \rho\colon G \to H.\end{aligned}$$

(The same homomorphism ρ can be used in both statements.)

Thus, in this example, the question of whether the action on the principal bundle can be diagonalized is precisely asking whether a homomorphism defined on Γ can be extended to be defined on all of G. That is the subject of the Margulis Superrigidity Theorem (A7.4).

Because we are considering measurable sections, it is natural to ignore sets of measure 0. Furthermore, a compact group can sometimes be an obstruction to finding a totally ρ-simple section (just as there is a compact group in the conclusion of the Margulis Superrigidity Theorem). These considerations lead to the following useful notion:

(6.7) **Definition** (weaker). A section $s\colon M \to P$ is *ρ-simple* if there is a compact subgroup C of H that centralizes $\rho(G)$, such that, for every $g \in G$ and almost every $m \in M$, we have

$$g \cdot s(m) \in s(gm) \cdot \rho(g) \cdot C.$$

In other words, there is some $c(g, m) \in C$, such that

$$g \cdot s(m) = s(gm) \cdot \rho(g) \cdot c(g, m).$$

We think of $c(g, m)$ as being "compact noise," that is, a small deviation from the ideal result.

(6.8) **Theorem** (Zimmer Cocycle Superrigidity Theorem). *Assume*

- *G acts ergodically on M, with finite invariant measure,*

- *the action lifts to a principal H-bundle P over M, where H is an algebraic group*[1]*, and*
- *G is simple, with $\mathbb{R}\text{-rank}(G) \geq 2$.*

Then, after passing to finite covers of G and M, there is a measurable ρ-simple section s, for some continuous homomorphism $\rho\colon G \to H$.

(6.9) *Remark.*

1) In the case where $P = P_\rho$ is the principal bundle defined from a homomorphism $\rho\colon \Gamma \to H$ as in Example 6.6, the Cocycle Superrigidity Theorem reduces to the Margulis Superrigidity Theorem (A7.4).
2) There is an analogue of the Cocycle Superrigidity Theorem for Γ-actions, where Γ is any lattice in G.
3) The Cocycle Superrigidity Theorem yields a section that is only known to be measurable, not continuous. The lack of smoothness is a shortcoming in applications to geometric questions.
4) In the terminology of cocycles, the Cocycle Superrigidity Theorem shows that if a cocycle satisfies certain mild hypotheses, then it is cohomologous to a homomorphism times a "noise" cocycle with precompact image.
5) For homomorphisms of lattices, the Margulis Superrigidity Theorem gives us a description of the homomorphisms into compact groups (in terms of field automorphisms of $\mathbb{C}$), not only those with noncompact image (see Corollary A7.7). Unfortunately, for a general action on a principal bundle, we do not have an analogous description of the cocycles into compact groups (the "compact noise"). The main obstacle to understanding such a cocycle is the question of whether or not it is cohomologous to a cocycle with countable image.

6B. Application to connection-preserving actions

Recall that an action of Γ on M is *finite* if the kernel of the action is a finite-index subgroup of Γ (see Definition 2.1).

(6.10) **Conjecture.** *Suppose Γ acts on a manifold M, and $\dim M$ is less than a certain function of the dimension of the smallest nontrivial representation of G. Then the Γ-action is finite.*

We have already verified a special case (see Theorem 2.19):

(6.11) **Theorem** (Farb-Shalen). *Conjecture 6.10 is true in the C^ω category for certain lattices when $\dim M = 2$.*

Here is an example of another setting where one can prove the conclusion of the conjecture:

(6.12) **Theorem.** *Conjecture 6.10 is true for actions that have both an invariant connection and a finite invariant measure.*

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[1]More precisely, H is a real algebraic group, or, more generally, the k-points of a k-group, where k is a local field of characteristic 0.

PROOF. *Step 1. There is a finite* Γ*-invariant measure on* P. Since $\dim M$ is small, the homomorphism ρ in the conclusion of the Cocycle Superrigidity Theorem is trivial. Thus, there is a compact subgroup C of H, such that, for every $\gamma \in \Gamma$ and $m \in M$, we have $s(\gamma m) \in \gamma \cdot s(m)C$. Hence, $s(M)C$ is a Γ-invariant subset of P.

Now, the desired conclusion is obvious if C is trivial, because then s is a Γ-equivariant embedding of M into P, so the invariant measure on M yields an invariant measure on P. In general, s is a non-equivariant embedding of M into P, so the measure on M yields a non-invariant measure on P. However, averaging the translates of this measure by elements of C yields an invariant measure.

Step 2. Γ *preserves a Riemannian metric on* P. Because there is a Γ-invariant connection, the desired conclusion follows from the simple fact that any connection on M defines a natural Riemannian metric on P.

Step 3. Completion of proof. The assertions of Steps 1 and 2 remain true when Γ is replaced with its closure $\overline{\Gamma}$ in $\mathrm{Diff}^1(M)$. This implies that $\overline{\Gamma}$ is a compact Lie group. Then, because there is obviously a homomorphism from Γ to $\overline{\Gamma}$, the Margulis Superrigidity Theorem (A7.4) implies that $\dim \overline{\Gamma}$ is "large." Because there is an explicit bound on the dimension of the isometry group of a Riemannian manifold in terms of the dimension of the manifold, we conclude that $\dim M$ is "large." □

The functions we have called "large" in the above proof are explicit and quite reasonable. For example, the following conjecture is true for connection-preserving actions.

(6.13) **Conjecture.** *If* $\dim M < n$*, then any volume-preserving action of* $\mathrm{SL}(n, \mathbb{Z})$ *on* M *is finite.*

Note that the conjecture is sharp, because $\mathrm{SL}(n, \mathbb{Z})$ acts on $\mathbb{T}^n$. Furthermore, we know all the connection-preserving actions of minimal dimension:

(6.14) **Theorem** (Feres, Zimmer). *Suppose* $\mathrm{SL}(n, \mathbb{Z})$ *acts* C^∞ *on a compact manifold* M *of dimension* n*, preserving a connection and a volume form. If* $n \geq 3$*, then* $M = \mathbb{T}^n$ *with the standard connection, and the action is the obvious one* (*or its contragredient*)*.*

SKETCH OF PROOF. The Cocycle Superrigidity Theorem gives us a homomorphism $\rho\colon \mathrm{SL}(n, \mathbb{R}) \to \mathrm{SL}(n, \mathbb{R})$. If ρ is trivial, then the proof of Theorem 6.12 applies. Thus, we may now assume ρ is the identity map (or the contragredient). Therefore, the action on P is measurably isomorphic to the standard action. Use the connection to obtain a smooth isomorphism from the measurable one. □

6C. From measurability to smoothness

The proof of Theorem 6.12 illustrates the use of a geometric structure to turn a measurable object into something better. In that case, the geometric structure is a connection. Another useful structure is hyperbolicity, which has been studied in this context by many people (e.g., S. Hurder, A. Katok, J. Lewis, R. J. Zimmer, R. Spatzier, E. Goetze, R. Feres, C. B. Yue, N. Qian, G. A. Margulis), D. Fisher, K. Whyte, and B. Schmidt. There are also situations in which Sobolev space techniques can be used to improve regularity of the measurable object.

Another approach to getting a conclusion that is better than measurable would be to modify the proof of the Cocycle Superrigidity Theorem itself to get a nicer section in some settings. The proof of the Margulis Superrigidity Theorem starts by using the Multiplicative Ergodic Theorem (or amenability, or, in Furstenberg's approach, random products) to get a measurable map. Then machinery makes the map better. All ways (such as the Multiplicative Ergodic Theorem, amenability, and random products) to create a map from nothing seem to give measurable maps, because those are the easiest to construct.

The Katok-Lewis and Benveniste examples of §1B(b) show that one cannot expect to get a smooth section in all cases. However, in some reasonable generality, one can hope to get a smooth section on an open set.

Comments

See [**3**] or [**4**] for a less brief introduction to cocycle superrigidity and its many geometric applications.

The Cocycle Superrigidity Theorem was first proved in [**12**], under the assumption that H is simple and the cocycle is "Zariski dense" in H. (A detailed exposition of the proof appears in [**13**, Thm. 5.2.5].) These restrictions on H were removed in [**17**]. The proofs are modeled on ideas in the proof of the Margulis Superrigidity Theorem. A recent generalization that applies to some groups of real rank one, and is proved by quite different methods, appears in [**7**]. (See [**1**] for the first positive results on cocycle superrigidity in real rank one.)

Regarding Remark 6.9(5), there exist cocycles that are not cohomologous to a cocycle with countable image [**11**, Rem. 2.5].

Theorem 6.12 appears in [**15**, Thm. G] (in a more general form that replaces the connection with any geometric structure of "finite type").

Theorem 6.14 appears in [**2, 16**].

E. Goetze and R. Spatzier used hyperbolicity to obtain C^r smoothness in a cocycle superrigidity theorem [**8**] that is a key ingredient in the proof of a classification theorem [**9**]. Other examples of the use of hyperbolicity to obtain regularity include [**6, 10**]. See [**14**] for an example of the use of Sobolev techniques to pass from a measurable metric to a smooth one.

R. Feres and F. Labourie [**5**] have a geometric approach to the proof of cocycle superrigidity that represents progress in the direction requested in Section 6C. This approach is also employed in [**3**].

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LECTURE 7

Fundamental Groups II (Arithmetic Theory)

It was pointed out in Example 1.6 that if Λ is a lattice in a connected Lie group H that contains G, then G acts by translations on $M = H/\Lambda$. Hence, the fundamental group $\pi_1(M)$ can be a lattice in a connected Lie group that contains G.

Conversely, we will see in this lecture that, under certain natural hypotheses, $\pi_1(M)$ has to be almost exactly such a group — a lattice Λ in a connected Lie group H that contains G *locally* (i.e., the Lie algebra of H contains the Lie algebra of G). The proof uses the Cocycle Superrigidity Theorem in a crucial way, and requires an engaging hypothesis.

7A. The fundamental group is a lattice

The following equivalence relation equates the fundamental group of M with the fundamental group of any of its finite covers:

(7.1) **Definition.** Two abstract groups Λ_1 and Λ_2 are *commensurable* if some finite-index subgroup of Λ_1 is isomorphic to a finite-index subgroup of Λ_2.

(7.2) **Theorem** (Zimmer, Lubotzky-Zimmer). *Suppose G is simple and acts on M with finite invariant measure μ, and $\mathbb{R}$-rank$(G) \geq 2$. Assume $\pi_1(M)$ is infinite and linear (i.e., $\pi_1(M)$ is isomorphic to a subgroup of $\mathrm{GL}(n,\mathbb{R})$, for some n).*

1) *If the action is totally engaging, then $\pi_1(M)$ is commensurable to a lattice in some connected Lie group H that contains G locally.*
2) *If the action is topologically engaging, then $\pi_1(M)$ contains a subgroup that is isomorphic to a lattice in some connected Lie group H that contains G locally.*

See Theorem 7.9 for a more refined version of this result.

One main idea in the proof. We prove only (2). (If the action is totally engaging, further arguments are needed to show that Λ is itself a lattice, instead of just containing one.)

Let $H = \mathrm{GL}(n,\mathbb{R})$ and $\Lambda = \pi_1(M)$, so, by assumption, Λ is (isomorphic to) a subgroup of H. Applying the Cocycle Superrigidity Theorem (6.8) to the associated principal H-bundle

$$P = (\widetilde{M} \times H)/\Lambda$$

yields a section $s\colon M \to P$, a homomorphism $\rho\colon G \to H$, and a compact subgroup C of H, such that

$$g \cdot s(m) = s(gm) \cdot \rho(g) \cdot c(g,m).$$

Assume, for simplicity, that Λ is discrete, so the quotient $\Lambda\backslash H$ is a nice topological space. From the definition of P, there is an H-equivariant projection $q\colon P \to \Lambda\backslash H$.

Let $\varphi = q \circ s\colon M \to \Lambda\backslash H$. Noting that q is G-invariant (because the action of G is on the factor $\widetilde{M}$ of P, not on H), we have

$$\varphi(m) = \varphi(gm) \cdot \rho(g) \cdot c(g,m).$$

By taking inverses to replace the right H-action on $\Lambda\backslash H$ with the left H-action on H/Λ, we obtain a map $\psi\colon M \to H/\Lambda$ with

$$\psi(m) = c(g,m)^{-1} \cdot \rho(g)^{-1} \cdot \psi(gm);$$

i.e.,

$$\psi(gm) = \rho(g) \cdot c(g,m) \cdot \psi(m).$$

Recalling, from the statement of Theorem 6.8, that C centralizes $\rho(G)$, we may mod out C to get

$$\theta\colon M \to C\backslash H/\Lambda$$

with

$$\theta(gm) = \rho(g) \cdot \theta(m),$$

so θ is G-equivariant.

Therefore, $\theta_*(\mu)$ is a G-invariant measure on $C\backslash H/\Lambda$. (However, this measure may be singular with respect to the Haar measure on $C\backslash H/\Lambda$; the image of θ may be a null set.) Ratner's Theorem (A2.3) tells us there is a connected subgroup H' of H containing G, such that $\Lambda' = \Lambda \cap H'$ is a lattice in H', and (a translate of) $\theta_*(\mu)$ is the Haar measure on $C'\backslash H'/\Lambda'$. Therefore,

> Λ contains a lattice (namely, $\Lambda' = \Lambda \cap H'$) in a connected Lie group (namely, H') that contains G.

This establishes (2). □

(7.3) *Remark.* In the course of the proof, we assumed Λ is discrete in $\mathrm{GL}(n,\mathbb{R})$. In general, some standard algebraic methods are used to convert Λ to a discrete group.

Filling a gap in the proof. The above proof of Theorem 7.2(2) is incomplete, because we did not show that ρ is nontrivial. (If $\rho(G) = \{e\}$, then we only showed that $\Lambda \supset \{e\}$, which is obviously not an interesting conclusion.) This is where we use the engaging hypothesis.

Suppose ρ is trivial. Recall that this implies we have a section $s\colon M \to P$ with

$$g \cdot s(m) = s(gm) \cdot c(g,m),$$

so we have a G-equivariant map $M \to P/C$. This means we have a G-invariant reduction of P to C, so there is a G-invariant H-equivariant map $P \to H/C$. From the definition of P, we also have a G-invariant H-equivariant projection $P \to H/\Lambda$. Thus, we have a G-invariant H-equivariant map

$$\tau\colon P \to H/C \times H/\Lambda.$$

Because C is compact, the action of H on H/C is proper, so the action of H on $H/C \times H/\Lambda$ is also proper, hence tame. Then, since G is ergodic on M, we conclude that the image of τ is contained in a single H-orbit, which we may identify with $H/(\Lambda \cap hCh^{-1})$ for some $h \in H$. Because Λ is discrete and hCh^{-1} is compact, we know $\Lambda \cap hCh^{-1}$ is finite. Thus, τ is a G-invariant reduction of P to a finite subgroup of Λ. Since P comes from $\widetilde{M}$, then it is not hard to show that $\widetilde{M}$ also has a G-invariant reduction to a finite subgroup of Λ. This contradicts any engaging hypothesis. □

7B. Arithmeticity of representations of the fundamental group

Theorem 7.2 can be extended to provide a description of the linear representations of $\pi_1(M)$. Given a representation $\rho\colon \pi_1(M) \to \mathrm{GL}(n,\mathbb{C})$, the goal is to describe $\rho\big(\pi_1(M)\big)$ in terms of arithmetic groups.

(7.4) **Definition.** A subgroup H of $\mathrm{GL}(n,\mathbb{C})$ is a $\mathbb{Q}$-*group* if it is the set of zeros of some collection of polynomials $P(x_{i,j})$with coefficients in $\mathbb{Q}$.

(7.5) **Example.** $B = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$ is a $\mathbb{Q}$-group, because it is defined by the system of polynomial equations

$$x_{2,1} = 0,\ x_{3,1} = 0,\ x_{3,2} = 0.$$

(7.6) **Definition.** An *arithmetic group* is a group commensurable to $H \cap \mathrm{GL}(n,\mathbb{Z})$, for some $\mathbb{Q}$-group H. Equivalently, one may say that an arithmetic group is a group that is commensurable to a Zariski-closed subgroup of $\mathrm{GL}(n,\mathbb{Z})$ (where $\mathrm{GL}(n,\mathbb{Z})$ inherits a Zariski topology because it is a subset of the variety $\mathrm{GL}(n,\mathbb{C})$).

(7.7) **Definition.** Let $S = \{p_1, p_2, \ldots, p_r\}$ be a finite set of prime numbers, and $\mathbb{Z}_S = \mathbb{Z}[1/p_1, \ldots, 1/p_r]$.

1) An S-*arithmetic group* is commensurable to $H \cap \mathrm{GL}(n,\mathbb{Z}_S)$, for some $\mathbb{Q}$-group H.
2) An $\mathfrak{s}$-*arithmetic group* is commensurable to a subgroup Λ of $\mathrm{GL}(n,\mathbb{Z}_S) \cap H$ that contains $\mathrm{GL}(n,\mathbb{Z}) \cap H$, for some $\mathbb{Q}$-group H. That is,

$$\mathrm{GL}(n,\mathbb{Z}) \cap H \ \subset\ \Lambda\ \subset\ \mathrm{GL}(n,\mathbb{Z}_S) \cap H.$$

(7.8) *Remark.*

1) The Margulis Arithmeticity Theorem (A7.9) states that every irreducible lattice in any semisimple group of real rank at least two is an arithmetic group.
2) If $H = [H,H]$, then any $\mathfrak{s}$-arithmetic subgroup Λ of H is a lattice in some locally compact group H'. Namely, H' is the closure of $\Lambda H_{\mathbb{R}}$ in $H_{\mathbb{R}} \times \prod_{i=1}^{r} H_{\mathbb{Q}_{p_r}}$.
3) Under the assumption that H is semisimple, T. N. Venkataramana has shown that any $\mathfrak{s}$-arithmetic subgroups is actually S'-arithmetic, for some $S' \subset S$. However, we do not want to assume H is semisimple, because, even though G is semisimple, it can be embedded in a non-semisimple group H, and G acts on H/Λ.

The following improvement of Theorem 7.2 does not require our usual assumption that M is a manifold: M can be any second countable, metrizable topological space that satisfies the usual covering space theory.

(7.9) **Theorem** (Lubotzky-Zimmer). *Suppose G is simple and acts on M with finite invariant measure μ, and $\mathbb{R}$-rank$(G) \geq 2$. Let $\rho\colon \pi_1(M) \to \mathrm{GL}(n,\mathbb{C})$ be any representation of $\pi_1(M)$, such that $\Lambda = \rho\big(\pi_1(M)\big)$ is infinite.*

1) *If the action is totally engaging, then $\rho(\Lambda)$ is arithmetic (in some $\mathbb{Q}$-group H that contains G locally).*

2) *If the action is engaging, then* $\rho(\Lambda)$ *is* $\mathfrak{s}$*-arithmetic* (*in some* $\mathbb{Q}$*-group* H *that contains* G *locally*).

3) *If the action is topologically engaging, then* $\rho(\Lambda)$ *contains an arithmetic subgroup of some* $\mathbb{Q}$*-group* H *that contains* G *locally.*

(7.10) *Remark.*

1) The group H in the conclusion of Theorem 7.9 is closely related to the Zariski closure of $\rho(\Lambda)$, but it is not exactly the same group. They have the same Levi subgroup, and the same representation of the Levi subgroup on the Lie algebra, but their unipotent radicals may not be isomorphic.

2) The proof of Theorem 7.9 uses *many* representations of Λ: not only real representations, but also p-adic representations and representations over fields of positive characteristic. Even when M is a manifold, representations of positive characteristic are involved.

If M is not a manifold, it is easy to construct actions where the "fundamental group" is S-arithmetic, but not arithmetic. For example, there is an action of G on $(G \times \text{Building})/\Lambda$, where Λ is an S-arithmetic group. This action is engaging, but not totally engaging.

(7.11) **Open question.** *If* M *is a manifold, and we assume the action is engaging, can* $\pi_1(M)$ *really be* $\mathfrak{s}$*-arithmetic without being arithmetic?*

Comments

Theorem 7.2(2) was proved in [**8**]. Theorem 7.9 was proved in [**4**] (and implies Theorem 7.2(1)). As suggested by our proof of Theorem 7.2, the papers also show that the action on M admits a (measurable) G-equivariant map to the natural action on a double-coset space $C\backslash H/\Lambda$, where H is the $\mathbb{Q}$-group in the conclusion of Theorem 7.9, and Λ is an arithmetic subgroup. Analogues of these results for actions of a lattice Γ in G were proved by D. Fisher [**1**]. In some cases where Λ is nilpotent, it was shown by D. Fisher and K. Whyte [**2**] that C is trivial and the equivariant map to H/Λ can be chosen to be continuous.

An action of G on a double-coset space $C\backslash H/\Lambda$ is said to be *arithmetic*, and it is known [**3**] that if G acts on M with finite entropy, then the G-action has a (unique) maximal arithmetic quotient. Thus, the above-mentioned results provide a lower bound on this maximal arithmetic quotient.

These results on the fundamental group of M are similar in spirit to results [**5, 7**] on the arithmeticity of holonomy groups of foliations whose leaves are symmetric spaces.

Y. Shalom [**6**] obtained some very interesting restrictions on $\pi_1(M)$ even when G has rank 1.

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LECTURE 8

Locally Homogeneous Spaces

This lecture is not intended to be a survey of locally homogeneous spaces, but just an indication of how two techniques of ergodic theory can be applied to handle a number of particular directions in the subject.

8A. Statement of the problem

(8.1) **Definition.** Suppose M and X are spaces, and $\mathcal{G}$ is a pseudogroup of local homeomorphisms of X. We say that M is *locally modeled on* $(X, \mathcal{G})$ if there is an atlas of local homeomorphisms from open sets in M to open sets in X, such that the transition functions in X are elements of $\mathcal{G}$.

(8.2) **Example.** Let $X = \mathbb{R}^n$, and let $\mathcal{G}$ be the set of all local diffeomorphisms that preserve the $\mathbb{R}^k$-direction. Then a space modeled on $(X, \mathcal{G})$ is a foliated manifold. (More precisely, the manifold is n-dimensional, and the leaves of the foliation are k-dimensional.)

We will discuss the following basic case:

(8.3) **Assumption.**

- *$X = G/H$, where G is a Lie group and H is a closed subgroup, and*
- *$\mathcal{G}$ consists of the restrictions of elements of G to open sets in X.*

(8.4) **Definition.** If M is locally modeled on $(G/H, \mathcal{G})$, we say M is *locally homogeneous* and M is a *form* of G/H.

(8.5) **Example.** If $G = \mathrm{SO}(1, n)$ and $X = \mathbb{H}^n$, then the forms of X are the hyperbolic manifolds. (Note that the definition of a hyperbolic manifold is not in terms of curvature — it is a *theorem* that if M is a Riemannian manifold of constant curvature -1, then M is of this form.)

(8.6) **Example.** Let

- $G = \mathrm{Aff}(\mathbb{R}^n) \cong \mathrm{GL}(n, \mathbb{R}) \ltimes \mathbb{R}^n$, and
- $X = \mathrm{Aff}(\mathbb{R}^n) / \mathrm{GL}(n, \mathbb{R}) \cong \mathbb{R}^n$.

Then the forms of X are the affinely flat manifolds.

(8.7) **Problem.** *Given $X = G/H$, understand the forms of G/H.*

1) *For example, find all compact forms.*
2) *When does there exist a compact form?*

Here is one important way to construct compact forms:

(8.8) **Definition.** If there is a discrete subgroup Γ of G, such that

- Γ acts properly discontinuously on G/H, and

- $\Gamma\backslash G/H$ is compact,

then $M = \Gamma\backslash G/H$ is a compact form of G/H. This is called a *compact lattice form* (or a *complete form*).

Even when G/H is a symmetric space, it was once a question whether or not there always exists a compact lattice form, but this case was settled by Borel:

(8.9) **Theorem** (Borel). *If G is semisimple and H is compact, then G/H has a compact lattice form.*

(8.10) *Remark.*

1) When H is compact, it is easy to see that $\Gamma\backslash G/H$ is a compact lattice form iff Γ is a cocompact lattice in G.
2) On the other hand, when H is not compact, a lattice is never properly discontinuous on G/H (for example, this follows from the Moore Ergodicity Theorem (A3.3)), so we need to look for other kinds of discrete subgroups.

(8.11) *Remark.* Although there do exist examples of compact forms, the feeling among researchers in the area is that there are not many of them. That is, the known results suggest that if G is simple and H is a closed, noncompact subgroup, then it is unusual for G/H to have a compact lattice form, or even to have a compact form.

If H is compact, then all compact forms of G/H are compact lattice forms, but, in general, the compact forms can be a significantly larger class than the lattice forms, as is illustrated by the following example.

(8.12) **Example.** Let $G = \mathrm{GL}(k,\mathbb{R})$. It is not difficult to see that there is a cocompact lattice Γ in G (because G is essentially $\mathrm{SL}(k,\mathbb{R}) \times \mathbb{R}$, and each factor has a lattice). Let

$$M = G/\Gamma.$$

Thinking of G as an open subset of $\mathbb{R}^{k\times k}$ in the natural way, it is easy to see that multiplication by any element of Γ is a linear map, so it preserves the affine structure on $\mathbb{R}^{k\times k}$. Hence,

$$M \text{ is affinely flat.}$$

However, M is *not* a double-coset space of $\mathrm{Aff}(\mathbb{R}^{k\times k})$. Indeed, it is believed, for every n, that the fundamental group of a compact lattice form of $\mathbb{R}^n$ must have a solvable subgroup of finite index, but the fundamental group Γ in our example is very far from being solvable.

(8.13) **Conjecture** (L. Auslander). *If M is a compact lattice form for*

$$\mathrm{Aff}(\mathbb{R}^n)/\,\mathrm{GL}(n,\mathbb{R}) \cong (\mathbb{R}^n, \text{as an affine space}),$$

then $\pi_1(M)$ has a solvable subgroup of finite index.

(8.14) *Remark.* J. Milnor conjectured that the compactness assumption on M could be eliminated, but G. A. Margulis found a counterexample to this stronger conjecture: for $n \geq 3$, he showed there exist free subgroups of $\mathrm{Aff}(\mathbb{R}^n)$ that act properly discontinuously on $\mathbb{R}^n$.

(8.15) *Remark.* The plausibility of the conjecture, contrasted with the example, shows at least the potential for lattice forms and general forms to be very different. Therefore, when it has been shown that a homogeneous space G/H does not have any compact lattice forms, the existence of a general compact form remains an interesting question.

Because the existence of compact forms is not well understood in general, it is natural to consider special cases, such as homogeneous spaces G/H with both G and H simple. As an example, here is a question about two of the many possible ways to think of $\mathrm{SL}(k,\mathbb{R})$ as a subgroup of $\mathrm{SL}(n,\mathbb{R})$.

(8.16) **Problem.** *Embed* $H = \mathrm{SL}(k,\mathbb{R})$ *in* $G = \mathrm{SL}(n,\mathbb{R})$ (*with* $2 \le k < n$) *in one of two ways, either:*

1) *in the top left corner:* $\begin{bmatrix} \boxed{\mathrm{SL}(k,\mathbb{R})} & 0 \\ 0 & \mathrm{Id} \end{bmatrix}$*, or*

2) *as the image of an* irreducible *representation* $\rho\colon \mathrm{SL}(k,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{R})$.

Show G/H *does not have any compact forms.*

We will see how Ergodic Theory can be applied to study these two cases, but even these simple examples are not fully understood, and completely different ergodic-theoretic ideas have been used for the two different cases:

1) F. Labourie, S. Mozes, and R. J. Zimmer used superrigidity arguments in a series of papers.
2) G. A. Margulis and H. Oh used a nonergodicity theorem based on the rate of decay of matrix coefficients along certain subgroups.

The techniques are complementary, in that the method used in (2) does not apply to any examples of type (1).

8B. The use of cocycle superrigidity

(8.17) **Theorem** (Labourie-Mozes-Zimmer). *Embed* $H = \mathrm{SL}(k,\mathbb{R})$ *in* $G = \mathrm{SL}(n,\mathbb{R})$ *in the top left corner* (*as in Problem 8.16*(*1*))*, and assume* $n \ge 6$*. Then* G/H *has:*

1) *no compact forms if* $k \le n/2$*, and*
2) *no compact lattice forms if* $k \le n-3$*.*

Proof of (2). Suppose $M = \Gamma\backslash G/H$ is a compact lattice form, and let $L \cong \mathrm{SL}(n-k,\mathbb{R})$ be a subgroup of the centralizer $C_G(H)$. Note that:

- L acts on M (because L centralizes H), preserving a finite measure (because G, Γ, and H are unimodular, and $\Gamma\backslash G/H$ is compact).
- Furthermore, $\Gamma\backslash G$ is a principal H-bundle over M, and $\mathbb{R}\text{-rank}(L) \ge 2$ (because $n-k \ge 3$).

Hence, we are in the setting to apply Cocycle Superrigidity.

For simplicity, assume $k < n/2$. (This simplifies the proof, because, in this case, L is bigger than H, so there is no nontrivial homomorphism $L \to H$. The argument can be extended to eliminate this assumption.) Thus, the homomorphism $\rho\colon L \to H$ in the conclusion of the Cocycle Superrigidity Theorem (6.8) is trivial, so there is an L-invariant reduction of the principal bundle $\Gamma\backslash G$ to a compact subgroup $C \subset H$. That is,

$$\text{there is an } L\text{-invariant section } s\colon M \to \Gamma\backslash G/C.$$

Now take a tubular neighborhood of the section in the fiber direction: let H_0 be a set of nonzero, finite measure in H, with $CH_0 = H_0$, so $s(M)H_0$ is a G-invariant set of nonzero, finite measure in $\Gamma\backslash G/C$. By taking the characteristic function of this set, and pulling back to $\Gamma\backslash G$, we obtain an L-invariant L^2 function on $\Gamma\backslash G$.

So the natural unitary representation of G on $L^2(\Gamma\backslash G)$ has an L-invariant vector. Because L is noncompact, the Moore Ergodicity Theorem (A3.3) implies that there is a G-invariant vector in $L^2(\Gamma\backslash G)$, so Γ is a lattice in G. This is impossible, because $H = \mathrm{SL}(n-k,\mathbb{R})$ is not compact. □

(8.18) *Remark.* Y. Benoist and G. A. Margulis (independently) showed that the homogeneous space $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n-1,\mathbb{R})$ has no compact lattice form if n is odd. It seems that the case $\mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(2,\mathbb{R})$ is still open for non-lattice forms, and that the cases $\mathrm{SL}(4,\mathbb{R})/\mathrm{SL}(3,\mathbb{R})$ and $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n-2,\mathbb{R})$ are open even for lattice forms.

8C. The approach of Margulis and Oh

(8.19) **Definition.** Let $(\rho,\mathcal{H})$ be a unitary representation of G. For any nonzero vectors $v,w \in \mathcal{H}$, the function $G \to \mathbb{C}$ defined by $g \mapsto \langle\, \rho(g)v \mid w \,\rangle$ is a *matrix coefficient* of ρ. For simplicity, we always assume v and w are K-invariant, where K is a fixed maximal compact subgroup of G.

When G is a simple group with finite center, and ρ has no invariant vectors, the Howe-Moore Vanishing Theorem (A3.1) tells us that each matrix coefficient tends to 0 as $g \to \infty$. The *rate* of decay to 0 is an important subject of study.

(8.20) **Definition.** Fix a simple Lie group G with finite center, and a maximal compact subgroup K. We say a subgroup H is *tempered* in G if $\exists q \in L^1(H)$, such that for every unitary representation $(\rho,\mathcal{H})$ of G with no invariant vectors, and all K-invariant vectors $v,w \in \mathcal{H}$, we have

$$|\langle \rho(h)v \mid w\rangle| \le q(h)\,\|v\|\,\|w\|. \tag{8.21}$$

Some subgroups of G are tempered and some are not. (For example, G is not a tempered subgroup of itself.) H. Oh proved a theorem that, in many cases, tells us whether or not a subgroup is tempered.

(8.22) **Example** (Oh).

1) Embed $\mathrm{SL}(2,\mathbb{R}) \hookrightarrow \mathrm{SL}(n,\mathbb{R})$ via an irreducible representation. The image is a tempered subgroup of $\mathrm{SL}(n,\mathbb{R})$ iff $n \ge 4$.
2) The subgroup $\mathrm{SL}(k,\mathbb{R})$, embedded in the top left corner of $\mathrm{SL}(n,\mathbb{R})$, is not tempered.

The decay of matrix coefficients implies an ergodicity theorem (namely, the Moore Ergodicity Theorem), but Margulis used the tempered condition to prove a nonergodicity theorem:

(8.23) **Theorem** (Margulis). *Suppose G acts on a measure space X with an invariant measure that is σ-finite but* not *finite. If H is a tempered subgroup of G, then H is not ergodic on X.*

Here is a more precise topological statement of the result. (Note that the conclusion of the topological version clearly implies that H is not ergodic on X.)

(8.24) **Theorem** (Margulis). *Suppose G acts on a noncompact, locally compact space X, preserving a σ-finite Radon measure, such that there are no G-invariant subsets of nonzero, finite measure. If H is a tempered subgroup of G and C is a compact subset of X, then $HC \neq X$.*

Proof. To simplify the notation somewhat, let us assume $H = \{h^t\}$ is a one-parameter subgroup of G. Then, because $\bigcup_{t=0}^{1} Kh^tC$ is a compact subset of X, there is a positive, continuous function ϕ on X with compact support, such that

$$\phi(h^t c) \geq 1 \text{ for all } c \in C \text{ and all } t \in [0,1], \tag{8.25}$$

and, by averaging over K, we may assume that ϕ is K-invariant.

Suppose $HC = X$. (This will lead to a contradiction.) Then, by assumption, for each $x \in X$, there is some $T_x \in \mathbb{R}$, such that

$$h^{T_x}(x) \in C. \tag{8.26}$$

Fix some large $T \in \mathbb{R}^+$. Because $\bigcup_{t=-(T+1)}^{T+1} Kh^tC$ is compact, and X has infinite measure, there is some K-invariant continuous function ψ_T on X, such that

$$\|\psi_T\|_2 = 1, \tag{8.27}$$

$$0 \leq \psi_T(x) \leq 1 \text{ for all } x \in X, \tag{8.28}$$

and

$$|T_x| > T + 1 \text{ for all } x \text{ in the support of } \psi_T. \tag{8.29}$$

We have

$$\begin{aligned}
\|\phi\|_2 \int_{|t|>T} q(t)\,dt &\geq \int_X \int_{|t|>T} \phi(h^t x)\,\psi_T(x)\,dt\,dx && \text{((8.21) and (8.27))} \\
&\geq \int_X \int_0^1 \phi(h^{T_x+t} x)\,\psi_T(x)\,dt\,dx && \text{(8.29)} \\
&\geq \int_X \psi_T(x)\,dx && \text{((8.25) and (8.26))} \\
&\geq 1 && \text{((8.27) and (8.28)).}
\end{aligned}$$

However, because $q \in L^1(\mathbb{R})$, we know that $\lim_{T\to\infty} \int_{|t|>T} q(t)\,dt = 0$. This is a contradiction. □

(8.30) **Corollary.** *If H is a tempered, noncompact subgroup of G, then G/H has no compact lattice form.*

Proof. If $\Gamma\backslash G/H$ is compact, then there is a compact set $C \subset \Gamma\backslash G$ with $CH = \Gamma\backslash G$. This contradicts the conclusion of the theorem.

Combining this with Example 8.22(1) solves a special case of Problem 8.16(2):

(8.31) **Example.** Let $G = \mathrm{SL}(n,\mathbb{R})$, with $n \geq 4$, and let H be the image of an *irreducible* representation $\rho\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{R})$. Then G/H has no compact lattice form.

Comments

Recent surveys that describe work of Y. Benoist, T. Kobayashi, and others on the existence of compact lattice forms include [**4, 5**].

An earlier survey that includes some discussion of non-lattice forms is [**6**]. Work of A. Iozzi, H. Oh, and D. W. Morris on compact lattice forms of $\mathrm{SO}(2,n)/H$ and $\mathrm{SU}(2,n)/H$ (where H may not be reductive) is described in [**3**].

The Auslander Conjecture (8.13) remains open; see [**1**] for a survey. Another recent result is [**2**]. Margulis' counterexample to Milnor's Conjecture (8.14) appears in [**9**].

The use of cocycle superrigidity described in Section 8B was developed in [**13, 7, 8**]. It inspired a related result of Y. Shalom [**12**, Thm. 1.7] that replaces the semisimple group L with a more general nonamenable group.

Example 8.22 (and results that are much more general) appear in [**11**]. The other results of Section 8C appear in [**10**].

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LECTURE 9

Stationary Measures and Projective Quotients

Suppose G acts continuously on a metric space X. (We do not assume X is a manifold.) The results in almost all of the previous lectures have assumed there is a finite G-invariant measure on X, but we will now discuss what to do when there is no such measure.

(9.1) **Example.** Let Q be a parabolic subgroup of G, and let $M = G/Q$. Then G acts on M, with no finite invariant measure (unless $Q = G$).

As a substitute for finite invariant measures, Section 9A provides an introduction to *stationary* probability measures. These will allow us to address the following basic question in Section 9B:

(9.2) **Question.** *How close is a general action to being a combination of two basic types:*

1) *an action with finite invariant measure, and*
2) *the action on some G/Q?*

9A. Stationary measures

(9.3) **Assumption.** *In this chapter, μ always denotes an admissible probability measure on G, which means that*

- *μ is symmetric (i.e., $\mu(A^{-1}) = \mu(A)$, where $A^{-1} = \{\, a^{-1} \mid a \in A \,\}$),*
- *μ is absolutely continuous (with respect to Haar measure), and*
- $\operatorname{supp}(\mu)$ *generates G.*

(9.4) **Definition.**

1) For any measure ν on X, the *convolution of μ and ν* is the measure $\mu * \nu$ on X defined by
$$\mu * \nu = \int_G g_* \nu \, d\mu(g);$$
i.e., $(\mu * \nu)(A) = \int_G \nu(g^{-1}A)\, d\mu(g)$.
2) ν is called *μ-stationary* (or simply *stationary*) if $\mu * \nu = \nu$.

(9.5) **Proposition** (Basic properties of stationary measures).

1) *If X is compact, then there exists a stationary measure.*
2) *Every stationary measure is quasi-invariant.*
3) *Suppose K is a compact subgroup of G. If μ is K-invariant, then every stationary measure is K-invariant.*

PROOF. (1) The function $\nu \mapsto \mu * \nu$ is an affine map on the space of probability measures on X. The space of probability measures is a compact, convex set (in the weak* topology on $C(X)^*$), so the classical Kakutani-Markov Fixed-Point Theorem (A8.4) implies this affine map has a fixed point. Any fixed point is a stationary measure.

(2) Let ν be a stationary measure, and suppose $\nu(A) = 0$. Since ν is stationary with respect to any convolution power $\mu^{*n} = \mu * \mu * \cdots * \mu$,

a) we see that $\nu(g^{-1}A) = 0$, for a.e. $g \in \operatorname{supp}(\mu^{*n})$, and
b) it is not difficult to show that $g \mapsto \nu(g^{-1}A)$ is continuous (because μ^{*n} is absolutely continuous).

Therefore,

$$\nu(g^{-1}A) = 0, \text{ for every } g \in \operatorname{supp}(\mu^{*n}).$$

Since μ is admissible, we know $\operatorname{supp}(\mu)$ generates G; this implies $\bigcup_n \operatorname{supp}(\mu^{*n}) = G$. Therefore, $\nu(g^{-1}A) = 0$, for every $g \in G$, so ν is quasi-invariant.

(3) For $k \in K$, we have $k_*\nu = k_*(\mu * \nu) = (k_*\mu) * \nu = \mu * \nu = \nu$. □

In some cases, we can find all of the stationary measures:

(9.6) **Example.**

1) Suppose G is simple and $X = G/Q$ for a parabolic subgroup Q. If μ is K-invariant, then, since K is transitive on G/Q, Proposition 9.5 implies that the only stationary measure is the unique K-invariant probability measure on G/Q.
2) Suppose
   - a semisimple Lie group G acts on a compact space X,
   - P is a minimal parabolic subgroup of G, and
   - K is a maximal compact subgroup of G.

   Because P is amenable, there is a P-invariant probability measure λ on X (see Appendix A9). Then $\nu = \mathrm{Haar}_K * \lambda$ is μ-stationary, for any K-invariant admissible measure μ. (This works because Haar_K projects to the stationary measure on G/P.) The following theorem shows that this construction produces all of the stationary measures (if G is simple).

(9.7) **Theorem** (Furstenberg). *If G is a simple Lie group, and μ is K-invariant and admissible, then the map $\lambda \mapsto \mathrm{Haar}_K * \lambda$ is an affine bijection between the space of P-invariant probability measures on X and the (G, μ)-stationary measures on X.*

Thus, every stationary measure can be obtained by starting with a P-invariant measure, and averaging over K.

(9.8) *Remark.*

1) If we change μ, the set of μ-stationary measures will change, but it can be shown that the set of measure classes will not change (if G is a simple Lie group). Thus, there is a natural collection of measures ("natural" meaning that it depends only on G, not on the choice of μ), namely, the measure classes that contain a stationary measure. Unfortunately, it may not be obvious which measure classes are members of the collection, but Theorem 9.7 provides a characterization that is often useful.

2) Theorem 9.7 shows that, in the case where μ is K-invariant, the collection of stationary measures does not depend on the choice of the specific measure μ, but only the choice of the maximal compact subgroup K.

9B. Projective quotients

(9.9) **Example.** Suppose a parabolic subgroup Q of G acts on a compact space Y with finite invariant measure λ. Induce the action to G (as in §1B(a)): i.e., let $X = (G \times Y)/Q$, where Q acts on both G (by right translations) and Y. Then X is a G-equivariant bundle over G/Q with fiber Y. Because the base and fiber are compact, we know that X is compact. By combining the stationary measure $\nu_{G/Q}$ on the base with the invariant measure λ on the fibers, we obtain a quasi-invariant measure on X; this measure is stationary. One should note that, in this example, the fiber measures are invariant (that is, the maps from fiber to fiber are measure preserving) — all the non-invariance is happening on the base G/Q.

(9.10) **Problem.**

1) *Find conditions under which any X with a stationary measure is (up to measurable isomorphism, or up to C^∞ isomorphism) induced from a measure-preserving action of a parabolic subgroup.*
2) *For $\mathbb{R}$-rank$(G) \geq 2$, does every (X, ν) have some G/Q as a quotient (unless ν is invariant)? (We assume ν is stationary.) In other words, is (X, ν) induced from some action (not necessarily measure preserving) of some parabolic subgroup?*

(9.11) *Remark.* For any G, there are actions not of the type described in (1).

(9.12) *Remark.* Problem 9.10(2) is open, but here is a counterexample in rank one.

Let Γ be a cocompact lattice in $\mathrm{SL}(2,\mathbb{R})$, such that Γ has a homomorphism onto the free group F_3 on three generators. Then F_3 maps onto F_2, and F_2 embeds in $\mathrm{SL}(2,\mathbb{R})$ as a lattice. The composition is a homomorphism from Γ into $\mathrm{SL}(2,\mathbb{R})$ with a large kernel. By restricting the usual projective action of $\mathrm{SL}(2,\mathbb{R})$ on S^1, we obtain an action of Γ on S^1. Induce the Γ-action to an action of $\mathrm{SL}(2,\mathbb{R})$:

Let $X = \big(\mathrm{SL}(2,\mathbb{R}) \times S^1\big)/\Gamma$. Then X is a circle bundle over $\mathrm{SL}(2,\mathbb{R})/\Gamma$.

Suppose there is a measurable G-map $X \to G/Q$, where Q is a proper parabolic subgroup of G. (This will lead to a contradiction.) Because X, by definition, contains a Γ-invariant copy of S^1, there is a Γ-map $f\colon S^1 \to G/Q$. Let N be the kernel of the composite homomorphism $\Gamma \to F_2$, so N acts trivially on S^1. Then N acts trivially on G/Q a.e. (with respect to $f_*\nu$, where ν is the quasi-invariant measure on S^1), so N fixes some point in G/Q. Therefore, N is contained in a conjugate of Q. Because Q is solvable, we conclude that N is solvable. This contradicts the fact that infinite normal subgroups of lattices cannot be solvable (because, by the Borel Density Theorem (A6.1), a normal subgroup of a lattice is Zariski dense in G).

(9.13) **Definition.** Suppose G acts on X with stationary measure ν. Let λ be the corresponding P-invariant measure provided by Theorem 9.7. We say the G-action on X is *P-mixing* if the action of P on (X, λ) is mixing, i.e., if

$$\lambda(h_n A \cap B) \to \lambda(A)\,\lambda(B) \quad \text{as } h_n \to \infty \text{ in } P.$$

(9.14) **Theorem** (Nevo-Zimmer). *Suppose G acts on a compact space X with stationary measure ν. Assume*

1) $\mathbb{R}$-rank$(G) \geq 2$, *and*
2) *the action is P-mixing.*

Then there is a parabolic subgroup Q and a Q-space Y with finite invariant measure, such that X is the induced G-space.

(9.15) *Remark.* The conclusion can fail if either (1) or (2) is omitted.

9C. Furstenberg entropy

(9.16) **Definition** (Furstenberg entropy). Suppose G acts on the space X, with a μ-stationary measure ν. Define

$$h(\nu) = -\int_X \int_G \log \frac{dg_*\nu}{d\nu}(x)\, d\mu(g)\, d\nu(x).$$

(9.17) **Lemma** (Basic facts on Furstenberg entropy).

1) $h(\nu) \geq 0$.
2) $h(\nu) = 0$ *iff ν is G-invariant.*
3) *If X is induced from a finite measure-preserving action of a parabolic subgroup Q, then $h(X) = h(G/Q)$.*

(9.18) *Remark.* Lemma 9.17(3) is a special case of the following more general fact (but we do not need this generalization):

> *If $f\colon (X_1, \nu_1) \to (X_2, \nu_2)$ is G-equivariant, and v_1 and ν_2 are stationary, then $h(\nu_1) \geq h(\nu_2)$. Furthermore, $h(\nu_1) = h(\nu_2)$ iff there is a "relative invariant measure."*

Theorem 9.14 has the following immediate consequence:

(9.19) **Corollary.** *If $\mathbb{R}$-rank$(G) \geq 2$, then the set of Furstenberg entropies $h(X)$ of P-mixing actions of G is a finite set, namely*

$$\{\, h(G/Q) \mid Q \textit{ is a parabolic subgroup of } G \,\}.$$

(9.20) *Remark.* The corollary is false in real rank one, even with the P-mixing assumption.

If $Q_1 \subset Q_2$, then there is a G-map from G/Q_1 to G/Q_2, so $h(G/Q_1) > h(G/Q_2)$. Thus, we have the following corollary.

(9.21) **Corollary.** *Assume $\mathbb{R}$-rank$(G) \geq 2$. Then $\exists c > 0$ such that:*

1) *For every P-mixing action of G on X, if $h(X) < c$, then there is a Q-invariant probability measure on X, for some maximal parabolic subgroup Q.*
2) *There exist P-mixing actions of G whose Furstenberg entropy is nonzero, and less than c.*

This implies that if $\mathbb{R}$-rank(G) is large and the Furstenberg entropy of the action is small, then the entire invariant measure program can be applied to the action of a group $L \subset Q$ with $\mathbb{R}$-rank$(L) \geq 2$.

(9.22) **Example.** Let $G = \mathrm{SL}(n,\mathbb{R})$. Then any maximal parabolic subgroup Q is block upper-triangular of the form

$$Q = \begin{bmatrix} \boxed{*} & * \\ 0 & \boxed{*} \end{bmatrix},$$

so Q must contain a copy of $\mathrm{SL}\big(\lfloor n/2 \rfloor, \mathbb{R}\big)$. Thus, the theory of measure-preserving actions applies to the action of $\mathrm{SL}\big(\lfloor n/2 \rfloor, \mathbb{R}\big)$.

In particular, if the action of G on X is engaging, and $h(X) \le c$, then Theorem 7.2 implies that $\pi_1(X)$ contains an arithmetic subgroup of a group L, where $L \supset \mathrm{SL}\big(\lfloor n/2 \rfloor, \mathbb{R}\big)$. For larger and larger values of c, we obtain weaker and weaker conclusions, until we reach the point at which we obtain no conclusion.

Comments

See [**6**] for a more thorough exposition of this material and related results.

The basic theory of stationary measures is due to H. Furstenberg [**1, 2**]. In particular, Theorem 9.7 is [**2**, Thm. 2.1].

Theorem 9.14 was proved in [**3**] (along with Remarks 9.8(1) and 9.15). See [**4**] for a more detailed discussion of applications to Furstenberg entropy.

A positive answer to Problem 9.10(2) is given in [**5**].

The example in Remark 9.12 admits a G-invariant, rigid, analytic geometric structure (see [**7**, Thm. 1.2]).

The quotient map $X \to G/Q$ is usually not continuous. However, every smooth G-action with a stationary, ergodic measure of full support admits a unique maximal quotient of the form G/Q, with the quotient map smooth on an invariant, dense, open, conull set. It is shown in [**7**] that if the action is real analytic and preserves a rigid geometric structure of algebraic type, then this maximal quotient is non-trivial (unless the stationary measure is G-invariant).

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LECTURE 10

Orbit Equivalence

Roughly speaking, the action of a group Γ_1 on a space X_1 is "orbit equivalent" to the action of Γ_2 on X_2 if the orbit space X_1/Γ_1 is measurably isomorphic to the orbit space X_2/Γ_2 (a.e.). It is sometimes the case that actions of two very different groups are orbit equivalent, but, when Γ_1 and Γ_2 are lattices in certain simple Lie groups, theorems of A. Furman and D. Gaboriau provide mild conditions that imply Γ_1 is isomorphic to Γ_2.

10A. Definition and basic facts

(10.1) **Assumption.** *All actions are assumed to be ergodic and non-transitive, with a finite invariant measure.*

(10.2) *Remark.* Abusing terminology, we will say a map $\varphi\colon X_1 \to X_2$ is *measure preserving* if it becomes measure preserving when the measures on X_1 and X_2 are normalized to be probability measures.

(10.3) **Notation.** If G acts on X, we use $\operatorname{Rel}(X,G)$ to denote the equivalence relation on X defined by the G-orbits, so $\operatorname{Rel}(X,G) \subset X \times X$.

(10.4) **Definition.** Suppose (G_1, X_1) and (G_2, X_2) are actions.

1) The actions are *orbit equivalent* if there is a measure-preserving bijection
$$\varphi\colon X_1 \to X_2 \text{ with } \varphi(G_1\text{-orbit}) = G_2\text{-orbit (a.e.)},$$
i.e., if $\operatorname{Rel}(X_1, G_1) \cong \operatorname{Rel}(X_2, G_2)$ (a.e.).

2) The actions are *stably orbit equivalent* if there is a measure-preserving bijection
$$\varphi\colon X_1 \times I \to X_2 \times I \text{ with } \varphi\big((G_1\text{-orbit}) \times I\big) = (G_2\text{-orbit}) \times I \text{ (a.e.)},$$
where I is the closed unit interval $[0,1]$.

(10.5) **Lemma** (Basic facts).

1) *If the G_1-orbits and G_2-orbits are uncountable (a.e.), then stable orbit equivalence is the same as orbit equivalence.*

2) *If G_1 and G_2 are discrete, then the actions (G_1, X_1) and (G_2, X_2) are stably orbit equivalent iff there are subsets $A_1 \subset X_1$ and $A_2 \subset X_2$ of positive measure, such that*
$$\operatorname{Rel}(X_1, G_1)|_{A_1} \cong \operatorname{Rel}(X_2, G_2)|_{A_2}.$$
(*That is, after restricting to two sets of positive measure, we have an orbit equivalence.*)

3) *If a discrete subgroup Γ of G acts on X, then the Γ-action on X is stably orbit equivalent to the induced G-action.*

(10.6) *Remark.* A classical example similar to (3) is the construction of a flow built under a function. It is clear that the orbits of the flow are the same as the orbits of the discrete dynamical system, but with the points extended into intervals.

10B. Orbit-equivalence rigidity for higher-rank simple Lie groups

For amenable groups, the orbit structure tells us nothing about the group, and nothing about the action:

(10.7) **Theorem** (Ornstein-Weiss). *If G_1 and G_2 are infinite, discrete, amenable groups, then any action of G_1 is orbit equivalent to any action of G_2.*

The following theorem shows that the situation for simple Lie groups is exactly the opposite.

(10.8) **Theorem** (Zimmer). *Let G_1 and G_2 be noncompact, simple Lie groups, with $\mathbb{R}$-rank$(G_1) \geq 2$. If (G_1, X_1) is orbit equivalent to (G_2, X_2), then G_1 is locally isomorphic to G_2, and, up to a group automorphism, the actions are isomorphic.*

Proof. Suppose $\varphi\colon X_1 \to X_2$ with $\varphi(G_1\text{-orbit}) = G_2\text{-orbit}$. For any $g \in G_1$ and $x \in X_1$, the points $\varphi(gx)$ and $\varphi(x)$ are in the same G_2-orbit, so there is some $c(g,x) \in G_2$ with

$$\varphi(gx) = c(g,x) \cdot \varphi(x).$$

The resulting map $c\colon G_1 \times X_1 \to G_2$ is a cocycle, i.e., for all $g, h \in G_1$, we have

$$c(gh,x) = c(g,hx)\, c(h,x), \text{ for a.e. } x \in X_1.$$

Thus, if we assume, for simplicity, that G_1 and G_2 have trivial center, then the Cocycle Superrigidity Theorem (A5.2) tells us that the cocycle c is cohomologous to a homomorphism $\rho\colon G_1 \to G_2$. Since G_1 and G_2 are simple (with trivial center), it is not difficult to see that ρ is an isomorphism. After identifying G_1 with G_2 under this isomorphism, the actions are isomorphic. □

(10.9) **Question.** *What about real rank one?*

(10.10) *Remark.* Superrigidity arguments in real rank one (such as the proof of the cocycle superrigidity theorem mentioned on page 45) typically use some kind of integrability condition on the cocycle (e.g., that, for fixed g, the function $c(g,x)$ is an L^1 function of x). But the cocycle obtained from an orbit equivalence cannot be expected to have any regularity, beyond being measurable, so such an argument does not apply. Nevertheless, see Remark 10.12 below for a positive result.

10C. Orbit-equivalence rigidity for higher-rank lattice subgroups

The following observation can be obtained easily from Theorem 10.8, by inducing the actions to G_1 and G_2:

(10.11) **Corollary** (Zimmer). *Suppose Γ_1 and Γ_2 are lattices in two simple Lie groups G_1 and G_2, with $\mathbb{R}$-rank $G_1 \geq 2$. If (Γ_1, X_1) is (stably) orbit equivalent to (Γ_2, X_2), then G_1 is locally isomorphic to G_2.*

(10.12) *Remark.* It would be interesting to extend Corollary 10.11 to lattices in (some) groups of real rank one. A partial result is known: if (Γ_1, X_1) is orbit-equivalent to (Γ_2, X_2), and Γ_i is a lattice in $\mathrm{Sp}(1, n_i)$, for $i = 1, 2$, then $n_1 = n_2$. The proof employs von Neumann algebras, rather than cocycle superrigidity.

The conclusion only tells us that the ambient Lie groups G_1 and G_2 are isomorphic, not that the lattices themselves are isomorphic, but the following result shows that, in general, one cannot expect more, because the many different lattices in a single simple Lie group G all have actions that are stably orbit equivalent:

(10.13) **Proposition.** *If Γ and Λ are lattices in G, then the Γ-action on G/Λ is stably orbit equivalent to the Λ-action on G/Γ.*

PROOF. Induce to G: both yield the G-action on $G/\Gamma \times G/\Lambda$. Since the induced actions are isomorphic, hence stably orbit equivalent, the original actions are stably orbit equivalent. □

Here is an interesting special case:

(10.14) **Example.** Any two nonabelian free groups F_m and F_n have stably orbit equivalent actions, because both are lattices in $\mathrm{SL}(2,\mathbb{R})$.

Furthermore, the actions of two different lattices (in the same group) can sometimes be orbit equivalent:

(10.15) **Proposition.** *Let Γ and Λ be lattices in a simple Lie group G, and assume $\mathbb{R}\text{-rank}(G) \geq 2$. The Γ-action on G/Λ is orbit equivalent to the Λ-action on G/Γ (not just stably orbit equivalent) iff $\mathrm{vol}(G/\Gamma) = \mathrm{vol}(G/\Lambda)$.*

However, the above examples (actions of Γ on G/Λ) are the only reason for two different lattices having orbit-equivalent actions:

(10.16) **Theorem** (Furman). *Suppose Γ acts on (X, μ), where Γ is a lattice in a simple Lie group G, with $\mathbb{R}\text{-rank}(G) \geq 2$, and*

for every lattice Γ' in G, there is no measure-preserving,
Γ-equivariant map $X \to G/\Gamma'$.

If (Γ, X) is orbit equivalent to the action (Λ, Y) of some discrete group Λ on some space Y, then Γ is isomorphic to Λ, (modulo finite groups), and the actions are isomorphic (modulo finite groups).

(10.17) *Remark.* Note that Λ is not assumed to be a lattice in Furman's theorem. It is not even assumed to be a linear group — it can be *any* group.

The quotients of standard actions have been classified. From this, it is often obvious that there is no quotient of the form G/Γ', so we have the following corollary.

(10.18) **Corollary.** *The conclusion of the theorem is true for:*

1) *the natural action of $\mathrm{SL}(n,\mathbb{Z})$ on $\mathbb{T}^n$,*
2) *the action of Γ on K/H, arising from an embedding of Γ in a compact group K, and H is a closed subgroup, and*
3) *the action of Γ on H/Λ arising from an embedding of G as a proper subgroup of a simple group H, and Λ is a lattice in H.*

10D. Orbit-equivalence rigidity for free groups

Let R be a countable equivalence relation on X (i.e., each equivalence class is countable), and assume $\mu(X) = 1$. In our applications, we have $R = \mathrm{Rel}(X, \Gamma)$, where Γ is discrete.

(10.19) **Definition** (Cost of an equivalence relation).

1) Suppose Φ is a countable collection of measure-preserving maps $\varphi_i \colon A_i \to B_i$, where $A_i, B_i \subset X$, such that $\varphi_i(x) \sim_R x$ for a.e. x, and $\{A_i\}$ covers X (a.e.).

   (a) Define
   $$\operatorname{cost}(\Phi) = \sum_i \mu(A_i).$$
   (b) We say Φ *generates* R if R is (a.e.) the smallest equivalence relation that contains the graph of every φ_i.

2) Define
$$\operatorname{cost}(R) = \inf\{\, \operatorname{cost}(\Phi) \mid \Phi \text{ generates } R\,\}.$$

Note that $1 \le \operatorname{cost}(R) \le \operatorname{cost}(\Phi) \le \#\Phi$, for any Φ that generates R.

(10.20) **Proposition.** *If Γ is amenable, then* $\operatorname{cost}\big(\operatorname{Rel}(X,\Gamma)\big) = 1$.

Proof. Since (X,Γ) is orbit equivalent to a $\mathbb{Z}$-action (see Theorem 10.7), the equivalence relation is generated by the action of a single transformation, so we may take Φ to be a one-element set. □

Observe that Φ defines the structure of a graph on each equivalence class. The following result gives a condition, in terms of these graphs, that ensures a particular set Φ of transformations attains the infimum.

(10.21) **Theorem** (Gaboriau). *If the graph resulting from Φ on each equivalence class is a tree (so, in particular, the equivalence relation is "treeable" in the sense of S. Adams), then* $\operatorname{cost}(R) = \operatorname{cost}(\Phi)$.

(10.22) **Corollary.** *Let F_n be the free group on n generators. For a free action of F_n on any space X, we have* $\operatorname{cost}\big(\operatorname{Rel}(X, F_n)\big) = n$.

The following important result is in sharp contrast to the elementary observation of Example 10.14.

(10.23) **Corollary.** *Free groups F_m and F_n do not have free orbit-equivalent actions (unless $m = n$).*

(10.24) *Remark.* It can be shown that $\operatorname{cost}\big(\operatorname{Rel}(X,\Gamma)\big) = 1$ for any action of a lattice Γ in a simple group of higher rank.

10E. Relationship to quasi-isometries

(10.25) **Definition.** Let (X, d_1) and (Y, d_2) be locally compact metric spaces. A *quasi-isometry* is a map $\varphi \colon X \to Y$ (not necessarily continuous), such that there are positive constants C and K satisfying
$$\frac{1}{C}\, d(x_1,x_2) - K \le d\big(\varphi(x_1), \varphi(x_2)\big) \le C\, d(x_1,x_2) + K \quad \text{for all } x_1, x_2 \in X$$
and $\varphi(X)$ is K-dense in Y (i.e., the K-ball around $\varphi(X)$ is all of Y).

(10.26) *Remark.* A bijective map f is a quasi-isometry iff it is *coarse bi-Lipschitz*, meaning that if we ignore small length scales (i.e., distances less than some cut-off K_1), then both f and f^{-1} are Lipschitz. More precisely, there is a constant C_1, such that
$$d_2\big(f(x_1), f(x_2)\big) < C_1\, d_1(x_1,x_2) \text{ whenever } d(x_1,x_2) \ge K_1,$$

and similarly for f^{-1}.

(10.27) **Example.**

1) Let M be a compact manifold. For any two metrics ω_1 and ω_2 on M, the identity map is a quasi-isometry from $(\widetilde{M}, \widetilde{\omega}_1)$ to $(\widetilde{M}, \widetilde{\omega}_2)$, so $\widetilde{M}$ has a canonical metric up to quasi-isometry.
2) Let Γ be a finitely generated group. For each finite generating set F, the group Γ can be made into a metric space by identifying it with the vertices of the Cayley graph $\operatorname{Cay}(\Gamma; F)$. The resulting metric is not canonical, because it depends on the generating set, but it is well defined up to quasi-isometry.

(10.28) **Proposition.** *If M is any compact, Riemannian manifold, then the fundamental group $\pi_1(M)$ is quasi-isometric to the universal cover $\widetilde{M}$.*

(10.29) **Example.** Any two cocompact lattices in the same simple Lie group G are quasi-isometric, because they are fundamental groups of manifolds with the same universal cover, namely the symmetric space associated to G.

The situation is very different for lattices that are *not* cocompact:

(10.30) **Theorem** (Schwartz, Farb, Eskin). *For non-cocompact, irreducible lattices, quasi-isometry is the same as commensurability.*

(10.31) *Remark.* More precisely, Theorem 10.30 states that:

1) Non-cocompact, irreducible lattices in two different semisimple Lie groups (with no compact factors) are never quasi-isometric. (Here, by *different*, we mean "not locally isomorphic.")
2) Quasi-isometric non-cocompact, irreducible lattices in the same semisimple Lie group are commensurable.

Furthermore, although it was not stated in (10.30), any finitely generated group quasi-isometric to a non-cocompact lattice is isomorphic to it, modulo finite groups.

Here is a possible connection between quasi-isometry and stable orbit equivalence:

(10.32) **Problem.** *It is known that:*

1) *Γ and Λ are quasi-isometric iff there is an action of $\Gamma \times \Lambda$ on a locally compact space X, such that each of Γ and Λ acts properly discontinuously and cocompactly.*
2) *Γ and Λ have stably orbit equivalent actions iff there is an action of $\Gamma \times \Lambda$ on a measure space Y with σ-finite invariant measure μ, such that each of Γ and Λ acts properly with quotient of finite measure (i.e., Y/Γ and Y/Λ have finite measure).*

If one could find a σ-finite invariant measure on X (whose quotients by Γ and Λ are finite), there would be a direct connection between quasi-isometry and stable orbit equivalence.

Comments

See [**9, 23**] for lengthier surveys of orbit equivalence that discuss many of these same topics, plus recent developments.

The Bourbaki seminar [**24**] surveys exciting recent work of S. Popa (including [**19**]) that uses the theory of von Neumann algebras to establish a cocycle superrigidity theorem and orbit-equivalence results for a class of actions of many groups. An ergodic-theoretic approach to Popa's Cocycle Superrigidity Theorem is given in [**7**].

N. Monod and Y. Shalom [**16**] proved a variety of orbit-equivalence rigidity theorems for actions of countable groups with non-vanishing bounded cohomology $H_b^2\big(\Gamma, \ell^2(\Gamma)\big)$.

Theorem 10.7 appeared in [**18**]. See [**12**, Chap. 2] for an exposition of a generalization [**1**] to amenable equivalence relations that are not assumed *a priori* to come from a group action.

Theorem 10.8, and Corollary 10.11 appeared in [**27**]. Remark 10.12 appeared in [**2**].

Theorem 10.16, and Corollary 10.18 appeared in [**6**] (based on theory developed in [**5**]). See [**13**] for a survey of Y. Kida's similar results for actions of mapping class groups.

The notion of cost (Definition 10.19) was introduced by G. Levitt [**15**], who also pointed out Proposition 10.20. The other results of Section 10D are due to D. Gaboriau [**8**]. See [**12**, Chap. 3] for an exposition.

See [**4**] for a survey of the many results that led to Theorem 10.30 and a related theorem for cocompact lattices. (An alternate proof of the higher-rank case appears in [**3**].) The proofs show that if $G \neq \mathrm{SL}(2,\mathbb{R})$, then every quasi-isometry of an irreducible, noncocompact lattice in G is within a bounded distance of the conjugation by some element of G that commensurates the lattice. Analogous results have been proved for other classes of groups, such as S-arithmetic groups [**25, 26**], graphs of groups [**17**], and fundamental groups of hyperbolic piecewise manifolds [**14**].

The characterizations of quasi-isometry and stable orbit equivalence in (1) and (2) of Problem 10.32 are due to M. Gromov [**11**, 0.2.C_2' and 0.5.E]. The property in (2) is called *measure equivalence*; see [**10**] for an introduction to the subject (including a proof of the relation with stable orbit equivalence) and a discussion of recent developments.

Any two amenable groups are measure equivalent, but they need not be quasi-isometric. Conversely, two quasi-isometric groups need not be measure equivalent [**9**, p. 173], so Problem 10.32 does not always have a positive solution. Even so, understanding the measure theory of quasi-isometries remains an interesting open problem. Results of Y. Shalom [**22**] and R. Sauer [**20**] use the fact that an invariant measure is easy to find when Γ and Λ are amenable.

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Appendix: Background Material

The lectures assume familiarity with the basic theory of Lie Groups, Topology, and Differential Geometry. As an aid to the reader, this appendix provides precise statements of some of the additional material that is used in the lectures. Note that we usually do not state the results in their most general form, and our references are usually not to the primary sources.

A1. Lie groups

This material can be found in [**4**] (and other texts that discuss real reductive Lie groups).

(A1.1) **Definition.**

1) A connected Lie group is *simple* if and only if it has no nontrivial, connected, proper, normal subgroups.
2) A connected Lie group is *semisimple* if and only if it has no nontrivial, connected, solvable, normal subgroups.

(A1.2) **Definition.** Let G and H be connected Lie groups. We say G is *locally isomorphic* to H if the Lie algebra of G is isomorphic to the Lie algebra of H. (When G and H are connected, this is the same as saying that the universal cover of G is isomorphic to the universal cover of H.)

(A1.3) **Theorem** (Structure theorem). *Let G be a connected Lie group. Then:*

1) *$G = LR$, where L is a closed, connected, semisimple subgroup, and R is a closed, connected, solvable, normal subgroup.*
2) *G is semisimple if and only if G is locally isomorphic to a direct product of simple Lie groups.*

(A1.4) **Definition.** Let G be a connected, semisimple Lie group.

1) Any maximal solvable subalgebra of the complex Lie algebra $\mathfrak{g}\otimes\mathbb{C}$ is said to be *Borel*.
2) A Lie subalgebra of $\mathfrak{g}\otimes\mathbb{C}$ is *parabolic* if it contains a Borel subalgebra.
3) A Lie subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is *parabolic* if its complexification $\mathfrak{p}\otimes\mathbb{C}$ contains a Borel subalgebra of $\mathfrak{g}\otimes\mathbb{C}$.
4) A Lie subgroup P of G is if its Lie algebra $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$.

(A1.5) *Remark* (Alternative characterization of parabolic subgroups). Let G be a connected, semisimple Lie group with finite center. For any $a \in G$, such that $\operatorname{Ad} a$ is diagonalizable, let

$$P_a = \left\{\, g \in G \;\middle|\; \{\, a^n g a^{-n} \mid n \geq 0 \,\} \text{ is bounded} \,\right\}.$$

It is easy to see that P_a is a subgroup of G. Furthermore, P_a is parabolic, and every parabolic subgroup is of this form (for some Ad-diagonalizable a).

(A1.6) **Proposition.** *If P is any parabolic subgroup of G, then G/P is compact.*

In fact, if K is any maximal compact subgroup of G (and G has finite center), then K is transitive on G/P.

The theory of roots provides a classification of the parabolic subgroups of any given semisimple group G. In particular:

(A1.7) **Proposition.** *Any connected, semisimple Lie group has only finitely many parabolic subgroups, up to conjugacy.*

A2. Ergodic Theory

(A2.1) **Definition** [**9**]. Suppose G acts on M, and μ is a measure on M.

1) We say μ is *invariant* if, for every $g \in G$, we have $g_*\mu = \mu$. (I.e., $\mu(g^{-1}A) = \mu(A)$ for every $A \subset M$.)
2) We say μ is *quasi-invariant* if, for every $g \in G$, the measure $g_*\mu$ has the same null sets as μ. (I.e., $\mu(g^{-1}A) = 0$ iff $\mu(A) = 0$.)
3) A (quasi-)invariant measure μ is *ergodic* if every G-invariant, measurable set is either null or conull. (I.e., for every G-invariant measurable set A, either $\mu(A) = 0$ or $\mu(M \smallsetminus A) = 0$.) We may also say that the action of G is *ergodic* (with respect to μ).

(A2.2) **Example.**

1) Any smooth volume form on a manifold M defines a measure that is quasi-invariant under any group of diffeomorphisms of M.
2) In particular, if Λ is a closed subgroup of a Lie group H, then any subgroup of H acts by translations on H/Λ, and volume is quasi-invariant.

(A2.3) **Theorem** (Ratner's Measure-Classification Theorem [**6**]). *Let*

- *H be a Lie group,*
- *G be a closed, connected, semisimple subgroup of H, with no compact factors,*
- *Λ be any closed subgroup of H, and*
- *μ be an ergodic G-invariant probability measure on H/Λ.*

Then there is a closed, connected subgroup L of H, such that

1) *the support of μ is a single L-orbit in H/Λ,*
2) *μ is L-invariant, and*
3) *$G \subset L$.*

A3. Moore Ergodicity Theorem

These results can be found in [**9**, Chap. 2].

(A3.1) **Theorem** (Howe-Moore Vanishing Theorem). *Let*

- *G be a connected, simple Lie group with finite center,*
- *$(\rho, \mathcal{H})$ be a unitary representation of G,*
- *$\{g_n\}$ be a sequence of elements of G, such that $\|g_n\| \to \infty$, and*
- *$v_1, v_2 \in \mathcal{H}$.*

If $\langle v_1 \mid w\rangle = 0$, *for every* $\rho(G)$*-invariant vector* $w \in \mathcal{H}$, *then*

$$\langle \rho(g_n)v_1 \mid v_2\rangle \to 0 \quad \text{as } n \to \infty.$$

(A3.2) **Corollary** (Moore). *Let*

- G *be a connected, simple Lie group with finite center,*
- L *be a closed, noncompact subgroup of* G, *and*
- $(\rho, \mathcal{H})$ *be a unitary representation of* G.

Then every $\rho(L)$*-invariant vector in* $\mathcal{H}$ *is* $\rho(G)$*-invariant.*

Proof. Take $\{g_n\} \subset L$. If $v_1 = v_2$ is $\rho(L)$-invariant, then $\langle \rho(g_n)v_1 \mid v_2\rangle = \|v_1\|^2$ is independent of n. □

(A3.3) **Corollary** (Moore Ergodicity Theorem). *Let*

- G *and* L *be as in Corollary A3.2,*
- Γ *be a lattice in* G, *and*
- μ *be the* G*-invariant probability measure on* G/Γ.

Then L *is ergodic on* G/Γ.

Proof. It is clear that the characteristic function of any L-invariant subset of G/Γ is an L-invariant vector in the Hilbert space $L^2(G/\Gamma)$. □

(A3.4) **Corollary.** *In the situation of Corollary A3.3,* Γ *is ergodic on* G/L.

A4. Algebraic Groups

See [**8**, Chap. 3] for basic results on algebraic groups over $\mathbb{R}$ (and other local fields). The connection with ergodic theory (via tameness) is discussed in [**9**, §3.1]. (However, tame actions are called “smooth” there [**9**, Defn. 2.1.9].)

(A4.1) **Definition.**

- We use $\mathbb{R}[x_{1,1}, \ldots, x_{k,k}]$ to denote the set of real polynomials in the k^2 variables $\{\, x_{i,j} \mid 1 \le i, j \le k \,\}$.
- For any $Q \in \mathbb{R}[x_{1,1}, \ldots, x_{k,k}]$, and any $k \times k$ matrix g, we use $Q(g)$ to denote the value obtained by substituting the matrix entries $g_{i,j}$ into the variables $x_{i,j}$. For example:
  - If $Q = x_{1,1} + x_{2,2} + \cdots + x_{k,k}$, then $Q(g)$ is the trace of g.
  - If $Q = x_{1,1}x_{2,2} - x_{1,2}x_{2,1}$, then $Q(g)$ is the determinant of the first principal 2×2 minor of g.
- A subset H of $\mathrm{SL}(k, \mathbb{R})$ is said to be *Zariski closed* if there is a subset $\mathcal{Q}$ of $\mathbb{R}[x_{1,1}, \ldots, x_{k,k}]$, such that

  $$H = \{\, h \in \mathrm{SL}(k, \mathbb{R}) \mid Q(h) = 0, \ \forall Q \in \mathcal{Q} \,\}.$$

  In the special case where H is a sub*group* of $\mathrm{SL}(k, \mathbb{R})$, we may also say that H is an *algebraic group* (or, more precisely, a *real algebraic group*).
- The Zariski closed sets are the closed sets of a topology on $\mathrm{SL}(k, \mathbb{R})$; this is the *Zariski topology*.

(A4.2) **Definition.** Let G be a Zariski closed subgroup of $\mathrm{SL}(k,\mathbb{R})$. The natural action of G on $\mathbb{R}^k$ factors through to an action of G on the projective space $\mathbb{R}P^{k-1}$. Thus, there is a natural action of G on any G-invariant subset M of $\mathbb{R}P^{k-1}$; in this setting, we say that G acts *algebraically* on M.

(A4.3) **Proposition.** *If an algebraic group G acts algebraically on a subset M of $\mathbb{R}P^{k-1}$, then every G-orbit is locally closed. I.e., every G-orbit is the intersection of an open set and a closed set.*

(A4.4) **Corollary.** *Every algebraic action is tame, in the terminology of Definition 3.10.*

(A4.5) **Definition.** The zero sets of homogeneous polynomials in k variables define a Zariski topology on $\mathbb{R}P^{k-1}$. The intersection of any Zariski open set with any Zariski closed set is said to be a (quasi-projective) *algebraic variety*.

The following basic observation shows that homogeneous spaces of algebraic groups are varieties:

(A4.6) **Proposition** (Chevalley's Theorem). *If L is a Zariski closed subgroup of an algebraic group H, then, for some k, there exist*

1) *an embedding $H \hookrightarrow \mathrm{SL}(k,\mathbb{R})$, with Zariski closed image, and*
2) *an H-equivariant embedding $H/L \hookrightarrow \mathbb{R}P^{k-1}$ whose image is a variety.*

A5. Cocycle Superrigidity Theorem

(A5.1) **Definition.** Assume a Lie group G acts measurably on a space M, with quasi-invariant measure μ, and H is a Lie group.

1) A measurable function $\eta\colon G \times M \to H$ is a (Borel) *cocycle* if, for every $g_1, g_2 \in G$, we have
$$\eta(g_1 g_2, m) = \eta(g_1, g_2 m)\, \eta(g_2, x) \text{ for a.e. } m \in M.$$
2) Two cocycles $\eta_1, \eta_2\colon G \times M \to H$ are *cohomologous* if there is a measurable function $\varphi\colon M \to H$, such that, for every $g \in G$, we have
$$\eta_1(g,m) = \varphi(gm)^{-1}\, \eta_2(g,m)\, \varphi(m) \text{ for a.e. } m \in M.$$
Note that being cohomologous is an equivalence relation.

(A5.2) **Theorem** (Cocycle Superrigidity Theorem [**9**, Thm. 5.2.5]). *Assume:*

- *a connected, simple Lie group G acts on a space M, with invariant probability measure μ,*
- *H is a connected, noncompact, simple Lie group with trivial center,*
- *$\eta\colon G \times M \to H$ is a cocycle,*
- *η is not cohomologous to any cocycle with its values in a connected, proper subgroup of H, and*
- $\mathbb{R}$-rank $G \geq 2$ *and G has finite center.*

Then η is cohomologous to a homomorphism. That is, there is a continuous homomorphism $\rho\colon G \to H$, such that η is cohomologous to the cocycle $\rho'\colon G \times M \to H$, defined by $\rho'(g,m) = \rho(g)$.

A6. Borel Density Theorem

(A6.1) **Theorem** (Borel Density Theorem [**2**, Cor. 4]). *Let Γ be a lattice in a semisimple Lie group G, and assume G has no compact factors. If $\rho\colon G \to \mathrm{GL}(n,\mathbb{R})$ is any finite-dimensional representation of G, then $\rho(\Gamma)$ is Zariski dense in $\rho(G)$.*

(A6.2) **Theorem** (Generalized Borel Density Theorem [**2**, Lem. 3]). *Suppose G is a real algebraic group with no proper, normal, cocompact, algebraic subgroups.*

If G acts algebraically on a variety V, then every G-invariant probability measure on V is supported on the fixed-point set.

A7. Three theorems of G. A. Margulis on lattice subgroups

Detailed proofs of these theorems appear in [**5**] and [**9**].

(A7.1) **Definition.** Let Γ be a lattice in a connected, noncompact, semisimple Lie group G with finite center. We say Γ is *irreducible* if ΓN is dense in G, for every closed, noncompact, normal subgroup of G.

(A7.2) *Remark.* Assume G is linear, and has no compact factors. Then Γ is reducible iff Γ is commensurable to a direct product $\Gamma_1 \times \Gamma_2$, with Γ_1 and Γ_2 infinite.

(A7.3) **Theorem** (Margulis Normal Subgroup Theorem). *Let Γ be an irreducible lattice in a connected, semisimple Lie group G with finite center, and assume* $\mathbb{R}\text{-rank}\, G \geq 2$. *Then Γ is simple, modulo finite groups.*

More precisely, if N is any normal subgroup of Γ, then either

1) *N is finite, or*
2) *Γ/N is finite.*

(A7.4) **Theorem** (Margulis Superrigidity Theorem). *Let*

- *Γ be an irreducible lattice in a connected, semisimple Lie group G with finite center, such that $\mathbb{R}\text{-rank}\, G \geq 2$,*
- *$\rho\colon \Gamma \to \mathrm{GL}(k,\mathbb{R})$ be a finite-dimensional representation of Γ, and*
- *H be the identity component of $\overline{\overline{\rho(\Gamma)}}$.*

Then there are

1) *a finite-index subgroup Γ' of Γ,*
2) *a compact, normal subgroup C of H, and*
3) *a continuous homomorphism $\widehat{\rho}\colon G \to H/C$,*

such that $\rho(\gamma) \in \widehat{\rho}(\gamma)\, C$, for every $\gamma \in \Gamma'$.

(A7.5) *Remark.* If G/Γ is *not* compact, and H has trivial center, then the subgroup C in the conclusion of Theorem A7.4 is trivial.

(A7.6) **Notation.** Let σ be any Galois automorphism of $\mathbb{C}$ over $\mathbb{Q}$. The automorphism σ_* of $\mathrm{GL}(k,\mathbb{C})$ is obtained by applying σ to each entry of any matrix. I.e.,

$$\sigma_*\big((a_{i,j})\big) = \big(\sigma(a_{i,j})\big).$$

(A7.7) **Corollary.** *In the setting of Theorem A7.4, assume that H is simple and $H_{\mathbb{C}}$ has trivial center, where*

$H_{\mathbb{C}}$ *is the (unique) smallest complex Lie subgroup of* $\mathrm{GL}(n,\mathbb{C})$ *that contains* H.

Then there exist

1) *a finite-index subgroup* Γ' *of* Γ,
2) *a Galois automorphism* σ *of* $\mathbb{C}$ *over* $\mathbb{Q}$, *and*
3) *a continuous homomorphism* $\widehat{\rho}\colon G \to H_{\mathbb{C}}$,

such that $\sigma_*\big(\rho(\gamma)\big) = \widehat{\rho}(\gamma)$, *for every* $\gamma \in \Gamma'$.

(A7.8) **Corollary.** *Let* Γ, G, *and* ρ *be as in Theorem A7.4. If* G *has no nontrivial representation of dimension* $\le k$, *then* $\rho(\Gamma)$ *is finite.*

(A7.9) **Theorem** (Margulis Arithmeticity Theorem). *Suppose* Γ *is an irreducible lattice in a connected, noncompact, semisimple, linear Lie group* G, *such that* $\mathbb{R}\text{-rank}\, G \ge 2$. *Then there is a closed, semisimple subgroup* G' *of* $\mathrm{SL}(k,\mathbb{R})$, *for some* k, *such that* Γ *is commensurable with* $G' \cap \mathrm{SL}(k,\mathbb{Z})$.

A8. Fixed-point theorems

(A8.1) **Theorem** (Lefschetz Fixed-Point Formula [**3**, §36]). *Let*

- f *be a homeomorphism of a compact* n*-manifold* M, *and*
- $I_f = \sum_{k=0}^{n} (-1)^k \big(\text{trace of } f \text{ on the homology group } H_k(M;\mathbb{Q})\big)$.

If $I_f \neq 0$, *then* f *has at at least one fixed point in* M.

(A8.2) **Theorem** (Franks [**1**]). *Let* f *be a homeomorphism of the torus* $\mathbb{T}^2$. *If*

- f *is homotopic to the identity,*
- f *is area preserving, and*
- *the mean translation of* f *is* 0, *i.e.,*

$$\int_{[0,1]\times[0,1]} \big(\widetilde{f}(x) - x\big)\, dx = 0,$$

where $\widetilde{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ *is a continuous lift of* f *to the universal cover,*

then f *has at least one fixed point.*

(A8.3) **Corollary.** *Let* Γ *be an irreducible lattice in a connected, semisimple Lie group* G *with finite center, and assume* $\mathbb{R}\text{-rank}\, G \ge 2$. *For any area-preserving action of* Γ *on* $\mathbb{T}^2$, *there is a finite-index subgroup* Γ' *of* Γ, *such that every element of* Γ' *has a fixed point.*

(A8.4) **Theorem** (Kakutani-Markov Fixed-Point Theorem [**10**, §2.1]). *Let*

- A *be an abelian group of (continuous) affine maps on a Banach space* $\mathcal{B}$, *and*
- C *be a nonempty, compact, convex,* A*-invariant subset of* $\mathcal{B}$.

Then A *has at least one fixed point in* C.

A9. Amenable groups

The main results of this section (that is, all but some parts of Remark A9.3) can be found in [**9**, §4.1 and §7.2]. See [**7**] for a much more extensive treatment of the subject.

(A9.1) **Definition.** A Lie group H is *amenable* if the left regular representation of H on $L^2(H)$ has almost-invariant vectors. I.e., for every compact subset C of H, and every $\epsilon > 0$, there is a unit vector $v \in L^2(H)$, such that

$$\|cv - v\|_2 < \epsilon, \quad \text{for every } c \in C,$$

where $(cv)(x) = v(c^{-1}x)$ for $x \in H$.

(A9.2) **Proposition.** *If an amenable Lie group H acts continuously on a compact, metrizable topological space X, then there is an H-invariant probability measure on X.*

(A9.3) *Remark.* There are many characterizations of amenability. For example, the following are equivalent:

1) H is amenable.
2) If H acts continuously on a compact, metrizable topological space X, then there is an H-invariant probability measure on X.
3) If H acts continuously, by affine maps, on a Frechet space, and C is a nonempty, H-invariant, compact, convex subset, then H has at least one fixed point in C.
4) There is a left-invariant *mean* on the space $C_{\text{bdd}}(H)$ of all real-valued, continuous, bounded functions on H. (I.e., there is a left-invariant, positive linear functional that is nonzero on the constant function 1.)
5) There exist *Følner sets* in H. I.e., for every compact subset C of H and every $\epsilon > 0$, there is a subset F of H, with $0 < \mu(F) < \infty$, such that
$$\mu\big(F \triangle cF\big) < \epsilon\, \mu(F), \text{ for every } c \in C,$$
where μ is the left Haar measure on H.

It is easy to see from (2) that compact groups are amenable, and combining a version of the Kakutani-Markov Fixed-Point Theorem (A8.4) with (3) implies that abelian groups are amenable. More generally:

(A9.4) **Proposition.**

1) *If a Lie group H has a closed, solvable, normal subgroup R, such that H/R is compact, then H is amenable.*
2) *If P is any minimal parabolic subgroup of any connected, semisimple Lie group, then P is amenable.*

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Name Index

Abels, 58
Adams, 21, 22, 70
Albert, 38
Auslander, 54, 58

Bader, 22
Bekka, 30
Benoist, 38, 56, 58
Benveniste, 4–6, 30, 45
Bernard, 39
Borel, 2, 20, 34, 54, 63, 79, 81
Burger, 11, 13, 14, 45, 65, 73

Candel, 39
Chevalley, 78
Choquet-Bruhat, 39
Connes, 72
Corlette, 45
Cowling, 72

de Bartolomeis, 58
de la Harpe, 30
Deffaf, 22
Druţu, 73
Dunne, ix

Epstein, 23
Eskin, 71

Farb, ix, 6, 11–14, 43, 71, 73
Feldman, 72
Feres, 39, 44–46
Fisher, ix, 6, 14, 31, 44, 46, 50, 51
Foulon, 39
Frances, 22
Franks, 13, 14, 80, 81
Furman, ix, 67, 69, 73
Furstenberg, 45, 62, 64, 65, 81

Gaboriau, 67, 70, 72, 73
Ghys, 11–14
Goetze, 44, 45
Greene, 39
Gromov, 30, 33–35, 37–39, 72, 73

Handel, 13, 14
Hasselblatt, 6
Henle, 81
Hitchman, 46
Howe, 28, 56, 76
Hurder, 44, 46

Iozzi, 45, 58, 65, 73

Kakutani, 62, 80, 81
Katok, 4–6, 30, 44, 45
Kazez, 15
Kechris, 73
Kida, 72, 73
Knapp, 81
Kobayashi, 58
Kowalsky, 21, 22

Labourie, 6, 7, 45, 46, 55, 58
Lafont, 73
Laptev, 74
Lefschetz, 11, 13, 80
Levitt, 72, 73
Lewis, 4–6, 30, 44, 45
Lifschitz, 14
Lubotzky, 31, 47, 49, 51

Margulis, 2, 5, 7, 10, 35, 42–46, 49, 54–58, 79, 80, 82
Markov, 62, 80, 81
Masur, 14
Melnick, 22
Miller, 73
Milnor, 54, 58
Monod, 11, 13, 14, 72, 73
Moore, 23, 28, 29, 54, 56, 76, 77
Morris, 7, 14, 46, 58, 82
Mosher, 73
Mozes, 55, 58

Navas, 13, 14
Nevo, ix, 22, 39, 64–66
Niblo, 73
Nomizu, 38, 39

Oh, 55, 56, 58, 59

Ornstein, 68, 73

Papadopoulos, 73
Parwani, 14, 15
Pier, 31, 82
Platonov, 82
Polterovich, 13, 15
Popa, 72–74

Qian, 44
Quiroga-Barranco, 22, 23, 39, 51

Rapinchuk, 82
Ratner, 29, 48, 51, 76, 82
Rebelo, 13, 15
Reeb, 9, 10, 13, 15
Rivin, 23
Rohlin, 73
Roller, 73
Rourke, 23

Sageev, 73
Sauer, ix, 72, 74
Schachermayer, 15
Schmidt, ix, 44
Schwartz, 71, 74
Schweitzer, 15
Series, 23
Shalen, 11–14, 43
Shalom, 50, 51, 58, 59, 72–74
Sherman, 28
Soifer, 58
Spatzier, 44, 45
Stuck, 21–23

Thurston, 9, 10, 13, 15
Tricerri, 58

Uomini, 15

Vaes, 74
Valette, 30
Venkataramana, 49
Vesentini, 58

Weinberger, 14, 15
Weiss, 68, 72, 73
Whyte, 6, 44, 50, 51, 73
Witte, 10, 11, 15
Wortman, 74

Yau, 39, 46
Yoshino, 58
Yue, 44

Zeghib, 21–23
Zimmer, 7, 21, 23, 31, 39, 42, 44–47, 49, 51, 55, 58, 59, 64–66, 68, 72, 74, 82

Index

abelian group, 36, 80, 81
abuse of terminology, 26, 67
admissible measure, 61, 62
affinely flat manifold, 53, 54
algebraic
  action, 19, 78, 79
  group, 17, 18, 33, 41, 43, 77–79
  hull, 19–22, 33–37, 39, 46
    C^r, 19, 37
    kth order, 37
almost-invariant vector, 28, 81
amenable
  equivalence relation, 72
  group, 27–28, 30, 31, 45, 62, 68, 70, 72, 73, 80–82
analytic
  continuation, 38
  function, *see* real analytic
  set, 12, 13
Anosov flow, 26
arithmetic
  action, 50
  group, 1, 10, 49, 50, 65
atomic measure, 19
Auslander Conjecture, 54, 58
automorphism
  infinitesimal, 38
  local, 38
  of $\mathbb{C}$, 2, 43, 79, 80
$ax + b$ group, 21

Banach space, 80
big lattice, 11–13
blowing up, 4–5, 30
Borel subalgebra, 75
bounded cohomology, 11, 14, 72, 73

Cantor set, 12
Cartan geometry, 22
Cartan matrix of an inner product, 18
Cayley graph, 71
centralizer, 11, 12, 33–35, 37–39, 42, 55
circle S^1, 10–14
  bundle, 63
cocycle, 41, 45, 68, 78
  cohomologous, 41, 43, 45, 68, 78
  identity, 41
cohomology group $H^k(M;\mathbb{R})$, 21
commensurable groups, 47, 49, 71, 72, 79, 80
compact
  group, 2, 27, 33, 35, 42–44, 47, 48, 54, 56, 61–63, 69, 76, 79, 81
  lattice form, 54–58
complete form, *see* compact lattice form
connection (on a principal bundle), 30, 33–38, 43–46
  rigidity of, 34, 38
convex set, 62, 80, 81
convolution of measures, 61, 62
cost of an equivalence relation, 70, 72
covering space, 25, 29, 30, 49
  double, 30
  finite, *see* finite cover
  regular, 27
  universal, 25–38, 47, 48, 71, 75, 80

descending chain condition, 19, 26

engaging, 25–30, 47, 48, 50, 65
  geometrically, 38
  topologically, 26–27, 29–30, 33, 35, 38, 47, 50
  totally, 26, 29–30, 47, 49, 50
entropy
  finite, 50
  Furstenberg, *see* Furstenberg entropy
equivalence relation, 67, 69, 70, 78
  amenable, *see* amenable equivalence relation
ergodic, definition of, 76
Euler characteristic, 12–14

finite
  action, 9–14, 43, 44
  cover, 26, 27, 29, 43, 47
  orbit, 9–13
  quotient, 26

volume, 1, 27
fixed point, 4, 9–14, 20, 30, 38, 62, 63, 79–81
flow built under a function, 68
Følner set, 81
form of G/H, *see* locally homogeneous
fractional part, 25
frame bundle, 17, 18, 34, 36, 42
kth order, 34, 35, 37
Frechet space, 81
free group, 2, 4, 29, 54, 63, 69–70, 73
fundamental
domain, 25, 27
group $\pi_1(M)$, 4–6, 25–31, 33–35, 47–51, 54, 71, 72
Furstenberg entropy, 64

generates equivalence relation, 70
geometric structure, 17–22
almost rigid, 6
analytic, 30, 33, 34, 65
G-invariant, 1, 5, 6, 18, 20–21, 65
kth order, 36
of finite type, 45
rigid, 5, 6, 39, 65, 66
almost, 6
germ of a function, 35
gluing, 4–6, 25, 30
graph of groups, 72

H-structure on M, *see* geometric structure
Heisenberg group, 21
Hilbert space, 27, 77
hyperbolic manifold, 29, 53, 72
hyperbolicity, 44, 45

induced action, 3–4, 63, 64, 67, 69
integrability condition, 68
interval $[0, 1]$, 25, 67
irrational rotation, 11, 12
irreducible lattice, definition of, 79
isometry group, full, 21, 34, 44

k-jet, 35
Kazhdan's property (T), 13, 14, 28, 30

lattice subgroup, definition of, 1
linear action, 2
linear functional, 81
Lipschitz function, 70
locally
closed, 78
contained, 47
free action, 42
homogeneous, 53–55
compact, 53–55, 58
isomorphic Lie groups, definition of, 75
modeled on $(X, \mathcal{G})$, 53
rigid action, 5–6

Lorentz manifold, 21–23
low-dimensional manifold, 5–6, 9–14, 43, 44

mapping class group, 14, 72, 73
matrix coefficient, 56
decay of, 28, 55, 56
mean
on $C_{\mathrm{bdd}}(H)$, 81
translation, 80
measure
equivalence of groups, 72
invariant, definition of, 76
quasi-invariant, definition of, 76
Radon, 57
σ-finite, 56, 71
metabelian group, 12
metric
Lorentz, 21
pseudo-Riemannian, 17, 18, 35, 58
Riemannian, 34, 44
mixing action, 63
μ-stationary, *see* stationary measure

nilpotent group, 12, 15, 50
nonamenable group, 28, 58
normal subgroup, 2, 10, 21, 63, 75, 79, 81
normalizer, 38

orbit equivalent, 67–70
stably, 67–69, 71, 72

P-mixing, 63–64
parabolic
subalgebra, 75
subgroup, 3, 61–64, 75–76
maximal, 64, 65
minimal, 62, 81
piecewise manifold, 72
principal bundle, 17–21, 25, 35–37, 39, 41–43, 46, 55, 56
associated, 33, 34, 47
profinitely dense, 26
proper action, 19, 21, 27–30, 35, 48, 53, 54, 58, 71
measurably, 58
not, 21

$\mathbb{Q}$-group, 49, 50
$\mathbb{Q}$-rank, 10, 11, 14, 15
quasi-isometry, 70–72

rank one, *see* real rank one
real
analytic, 10–15, 30, 33, 34, 43, 65
rank
definition of, 2
one, 45, 50, 51, 59, 63, 64, 68, 74
reduction
of a principal bundle, 17, 19, 33, 36, 37
G-invariant, 18–20, 48, 56

of the universal cover, 25–29, 48
relative invariant measure, 64
representation
adjoint, 20, 21, 37, 38
finite-dimensional, 2, 5, 9, 10, 20, 21, 33–38, 43, 49, 50, 79, 80
irreducible, 55, 57
Gromov, 33
infinite-dimensional, 33
p-adic, 50
positive characteristic, 50
regular, 27, 28, 81
unitary, 28, 29, 51, 56, 59, 76, 77
irreducible, 56
residually finite, 30

S-arithmetic group, 49, 50, 72
$\mathfrak{s}$-arithmetic group, 49, 50
section of a bundle, 17, 20, 25, 26, 28–30, 37, 41–43, 45, 47, 48, 56
ρ-simple, 42, 43
totally, 41, 42
semisimple Lie group, definition of, 75
simple Lie group, definition of, 2, 75
$\mathrm{SL}(2,\mathbb{R})$, 2, 4, 21, 28, 56, 57, 63, 69, 72
$\mathrm{SL}(n,\mathbb{R})$, 1, 3, 4, 21, 35, 44, 45, 54–58, 65, 77, 78, 80
$\mathrm{SL}(n,\mathbb{Z})$, 4, 5, 10–15, 44, 69, 80
Sobolev space, 44, 45
solvable
discrete group, 54, 63
Lie group, 1, 75, 81
sphere
homology, 14, 15
S^2, 12
S^n, 4, 30
square $[0,1]^2$, 25
stabilizer
of a point, 3, 19–20, 22, 27, 42
of a vector, 29
stationary measure, 61–65

T_0 topological space, 18
tame
action, 18–19, 27, 29, 48, 77, 78
topological space, 18, 19
tangent bundle, 18, 34
kth order, 37
tempered subgroup, 56–57
Theorem
Borel Density, 2, 20, 34, 63, 79
Chevalley's, 78
Cocycle Superrigidity, 38, 41–45, 47, 55, 58, 68, 72, 78
Denjoy's, 11
Farb-Shalen, 13, 43
Franks Fixed-Point, 13, 80
Frobenius, 38
Ghys-Burger-Monod, 11, 13
Gromov's
Centralizer, 34, 35, 38
Open-Dense, 38
Howe-Moore Vanishing, 56, 76
Kakutani-Markov Fixed-Point, 62, 80, 81
Lefschetz Fixed-Point, 11, 13, 80
Margulis, 2, 5
Arithmeticity, 2, 49, 80
Normal Subgroup, 2, 10, 79
Superrigidity, 2, 10, 35, 42–45, 79–80
Moore Ergodicity, 29, 54, 56, 76–77
Multiplicative Ergodic, 45
Ratner's, 29, 48, 76
Reeb-Thurston Stability, 9, 10, 13
tiling, 25
topologically engaging, *see* engaging, topologically
torus
$\mathbb{T}^2$, 13, 15, 25, 80
$\mathbb{T}^n$, 4, 14, 44, 46, 69
totally engaging, *see* engaging, totally
treeable equivalence relation, 70
tubular neighborhood, 56

variety, algebraic, 19, 20, 49, 78–79
vector
bundle, 20, 35, 37, 42
field, 4, 5, 33, 34, 37–39
very big lattice, 12–13
von Neumann algebra, 68, 72

Zariski
closed, 17, 19, 20, 49, 77–78
closure, 19, 20, 33, 50, 79
dense, 1, 2, 45, 63, 79
topology, 49
definition of, 77